Whence LASSO? A Rational Interpretation

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Abstract

This paper rationalizes the LASSO algorithm based on uncertain fat-tail priors and max-min robust optimization. Our rationalization excludes heuristic learning or restrictive prior assumptions in the original interpretation of LASSO (Tibshirani (1996)). In our setting, economic agents (arbitrageurs) face ambiguity about fat-tail shocks and in equilibrium, they ignore a reasonable range of ambiguous signals but respond linearly to almost unambiguous signals. With this LASSO equivalent strategy, arbitrageurs can amass extra market power which induces a "cartel" to protect their aggregate profit from being competed away. This result shows a new mechanism for limited arbitrage.

Keywords: LASSO, fat tails, model risk, robust optimization, limits to arbitrage

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1 Introduction

Machine learning techniques have been widely used in economics and finance to make predictions, classifications, or decisions based on sample data (Jordan and Mitchell (2015), Athey (2018), Nagel (2021), and Hastie, Tibshirani, Friedman, and Friedman (2009)). Most machine learning tools were developed in the fields of statistics and computer science. They have been proven by extensive applications to be superior in prediction accuracy and computational efficiency. Nonetheless, there are two important economic questions to be addressed: (1) Are machine-learning methods rational choices for economic agents? (2) What would happen if machine learning becomes a new doctrine in financial markets? Answering these questions may generate new insights into topics in asset pricing and risk management.

In this paper, we rationalize a widely used machine learning algorithm, the *Least Absolute Shrinkage and Selection Operator* (LASSO), invented by the renowned statistician Robert Tibshirani in 1996. It is a linear regression with an l_1 norm penalty term in its loss function. This term is critical for the LASSO to achieve both variable selection and shrinkage. A simple explanation for the l_1-penalty is the argument of *Occam's Razor* or the principle of parsimony, akin to behavioral economics. A statistical interpretation proposed by Tibshirani (1996) is that the LASSO can be derived using the *maximum a posteriori* (MAP) estimate under a Laplace (double exponential) prior on the estimated parameter. Yet, this is not Bayesian rational. The MAP estimate is a heuristic learning rule as it uses the posterior *mode* as the point estimate, without integrating all the useful posterior information. Tibshirani's interpretation is also restrictive, as it only works with a pure and fixed Laplace prior.

As a concrete economic setting, we develop a model of arbitrage trading to demonstrate that the LASSO algorithm can be an equilibrium strategy chosen by Bayesian-rational agents (traders) when they have uncertain fat-tail priors (model risk). Our derivation of the LASSO strategy does not rely on a heuristic (MAP) learning rule or the restrictive assumption of a pure and fixed Laplace prior. Thus, our theory provides the first economic rationale for the LASSO algorithm. Our interpretation highlights the robustness of LASSO which stems from the endogenous inaction region chosen by agents to ignore directionally ambiguous signals. This sacrifices little optimality because agents still respond in a nearly optimal manner to the directional, most profitable signals. We also find that the robust LASSO strategy enables agents to amass extra market power that protects their aggregate profit even when their population approaches infinity. This is a novel channel for inefficient markets.

LASSO and its variants have been extensively used in financial studies. Rapach, Strauss, and Zhou (2013) apply the LASSO to study lead-lag relationships among monthly international stock returns. Goto and Xu (2015) use the graphical LASSO algorithm to solve a sparse estimator of the inverse covariance matrix in mean-variance portfolio optimization. Chinco, Clark-Joseph, and Ye (2019) apply the LASSO to select short-term predictors and forecast individual stock returns one-minute ahead, given cross-sectional returns over the past few minutes. Gu, Kelly, and Xiu (2020) apply a zoo of machine learning tools, including the LASSO, to study the time-series predictability of monthly individual stock returns. Freyberger, Neuhierl, and Weber (2020) apply the adaptive group LASSO to identify the relationships between numerous firm characteristics and cross-section of expected stock returns. Kozak, Nagel, and Santosh (2020) utilize the elastic net, a variant of LASSO, to construct a robust stochastic discount factor which integrate the explanatory power of a large number of cross-sectional return predictors. Dong, Li, Rapach, and Zhou (2022) employ a variety of LASSO-related shrinkage techniques to extract predictive signals from long-short anomaly portfolio returns in a high-dimensional setting. Huang and Shi (2022) apply the adaptive group LASSO to government bonds and construct a macro factor based on 30 predictors.

There are limited theoretical studies about LASSO in economics. Gabaix (2014) develops a sparsity model, in the spirit of LASSO, for the *anchoring-and-adjustment* bias (Tversky and Kahneman (1974)) and the limited attention (Sims (2003)). Martin and Nagel (2022) develop an equilibrium model to study cross-sectional return predictability due to sparsity or shrinkage when investors face a high-dimensional prediction problem. Both papers view the LASSO as a result of bounded rationality, agreeing with Tibshirani's interpretation.

The MAP method uses the posterior mode (instead of the mean) as the point estimate. It can lose applicability when uncertainty about the prior parameter(s) arises. In general, a rational agent will not directly use the MAP estimate as it overlooks valuable information. Tibshirani's interpretation of LASSO requires the prior to be a pure and fixed Laplace distribution which has a sharp peak at the origin and enables the MAP learning rule to generate sparse solutions. Yet this prior assumption is often unverified in real applications. It is unclear how relevant the Laplace prior is in many applications of LASSO. If this prior is just an approximation, it is still unclear how good the approximation is or whether model uncertainty matters. The Laplace distribution is found useful to fit the data of stock returns.[1] It is difficult to justify that the Laplace prior holds in general, especially when our prior knowledge is vague. If the validity of LASSO hinges on a fixed and pure Laplace prior, then it may work under certain circumstances but cause unexpected problems in other conditions.

The fact that the MAP learning is heuristic does not exclude the possibility of a Bayesian rational explanation of LASSO. Why is it important to look for an economic rationale for a simple algorithm? As a humble answer, it is at least pedagogical. There is no such theoretical work in the literature, perhaps for two reasons:

[1]See Mantegna and Stanley (1999), Lillo and Mantegna (2000), Silva, Prange, and Yakovenko (2004)

1. The MAP-based interpretation theoretizes the LASSO as a scientific tool with Bayesian logic and hence lend support to its *mathematical* legitimacy. This is undoubtedly the merit of Tibshrina's contribution. As a result, LASSO users may not examine its *economic* legitimacy and some fundamental issues may have been shadowed by the glory of LASSO's power. This is not an unusual example. We have been endowed with a large library of machine learning tools. The literature also gives us abundant statistical knowledge about them. Nonetheless, we seem to have only meager understanding as to whether and why those tools are economically sensible to begin with. A similar phenomenon, as noted by McQueen and Vorkink (2004), is in the literature of statistical models of volatility clustering, such as the *autoregressive conditional heteroskedasticity* (ARCH) model (Engle (1982)) and the generalized ARCH model (Bollerslev (1986)). The popularity of using these models stems from their power of fitting the data. Yet, "our theoretical knowledge of why volatility clusters is paltry".

2. Different fields have developed different systems of topics, methods, and standards. For example, the MAP estimate is taught in most statistics and machine learning textbooks but rarely mentioned in economics or finance textbooks. This kind of difference can delay scientific discoveries and call for interdisciplinary efforts. As another implication, the financial industry has hired many quants with strong training in math, physics, and engineering. The similar background may reinforce the quant mindset and sometimes hamper economic thinking. This may build up systemic risks, as probably exemplified by the *quant meltdown* in 2007 (Khandani and Lo (2011), Mussalli (2018)).

That is why we need theoretical research at the interface between machine learning and financial economics. The emerging literature leans toward a behavioral perspective (Gabaix (2014), Mullainathan and Spiess (2017), and Camerer (2018)). This does not preclude the possibility of a rational theory. For example, can the LASSO method ever be an equilibrium strategy? If no, then there should exist profitable deviations which may lead to improvements of the LASSO. If yes, then this sounds new and needs to be formally addressed.

Our theory proves the rational choice of the LASSO in an economically meaningful setup. We propose general questions at the beginning, but to address those questions, we need a concrete setting to specify the objective functions of agents and define the equilibrium. This approach has several benefits. First, the general audience can have a better understanding of the reasoning and mechanism when the economic setting is clearly defined. We can deliver generic intuitions through specified models. Moreover, this allows us to perform normative analysis of such issues as the performance of strategies, market stability, and price efficiency.

Our insight is based on the robustness of LASSO. The machine learning literature treated robustness as a secondary property of LASSO (Hastie, Tibshirani, and Wainwright (2015)). It can be "theoretized" by altering the loss function for the estimation problem (Xu, Caramanis, and Mannor (2008)). This approach is widely adopted in statistics but not economically grounded. For example, the MAP rule can be "theoretized" by the "hit-or-miss" loss function (Robert et al. (2007) [p. 166]), but this does not *rationalize* it. To prove an economic rationale for a machine-learning method, we have to formulate standard utility functions that agents optimize, rather than twisting the loss function to alter their learning rules.

The robustness of LASSO stems from its finite inaction zone which may drop ambiguous signals and keep unambiguous ones. This can be an economic reason for using the LASSO: if agents are uncertain about the direction of some fat-tailed signals, they may only respond to strong signals and ignore vague ones. This can avoid betting on the wrong side and suffering from fat-tailed losses. To formalize our intuition, we develop a model that incorporates two related features: uncertain fat-tail priors and robust optimization. Both issues are highly relevant to the trading of arbitrageurs in financial markets.

Specifically, we design a trading model where ambiguity-averse arbitrageurs predict and exploit pricing errors caused by random fat-tail shocks. We use a general Gaussian-Laplacian mixture distribution for the stock value. With a linear pricing rule,[2] random fat-tail shocks can produce disproportionate pricing errors, the frequency and magnitude of which are tuned by the mixing weight and the fat-tail scale parameter, respectively. Arbitrageurs are uncertain about the scale parameter. Each of them makes robust trading decisions by optimizing the *max-min expected utility*, as axiomatized by Gilboa and Schmeidler (1989).

We show that the equilibrium robust strategy chosen by arbitrageurs is equivalent to the LASSO estimate of pricing errors, conditional on the order flow (or the price change) observed in a time window right before their trading. Specifically, the strategy has an endogenous threshold inversely related to the averaged scale of fat-tail shocks but independent of their frequency, whereas the response intensity beyond the inaction zone is proportional to the frequency but independent of the scale parameter. Thus, under fairly general conditions, we show that the use of LASSO is a Bayesian rational strategy optimally chosen by agents who are concerned about fat-tailed model risks. This economic interpretation does not use any heuristic learning rule, nor make the restrictive assumption of a pure and fixed Laplace prior, which is an extreme case of our assumed general mixture distribution.

[2]The empirical price impact function, which measures the average price change in response to the size of an incoming order, is sublinear with some concavity. See Loeb (1983), Grinold and Kahn (2000) [p. 453], Gabaix, Gopikrishnan, Plerou, and Stanley (2006), and Kyle and Obizhaeva (2016). The linear pricing rule can be endogenized in multi-period Kyle-type models (e.g., Kyle (1985), Holden and Subrahmanyam (1992), Foster and Viswanathan (1994, 1996)) by assuming that market makers adhere to the Gaussian belief or restrict their considerations to linear pricing strategies perhaps for simplicity and robustness.

We also find that the robust LASSO strategy can outperform the optimal benchmark strategy, a nonlinear smooth response, which ignores the model risk and optimizes the subjective expected utility.[3] The benchmark strategy is highly susceptible to the bias of estimation and to the competition among traders. This easily loses profits if the estimate deviates from the true prior or if the number of competitors increases. In contrast, the performance of the LASSO strategy is robust to the estimate bias because its inaction zone avoids small but frequent mistakes. More importantly, its performance is much less sensitive to traders' competition. Even as the number of arbitrageurs goes to infinity, their aggregate profit does not vanish but converges to a positive level. This is a "cartel" effect induced by the under-trading (shrinkage) of the LASSO strategy. Its conservativeness mitigates traders' competition, allowing them to accumulate extra market power to protect their aggregate profit from being competed away. Consequently, even an infinite number of them can act as if a monopolist buys or sell the asset at a better price than the fair one. This seemingly collusive behavior does not involve any trading or financial constraints, nor require any communication device or explicit agreement. The "cartel" is facilitated tacitly by traders' uncoordinated exercise of risk management. Therefore, our results reveal a novel channel for limits to arbitrage.

Our two-period model describes a market with *short-lived* and *infrequent* return predictability, consistent with the empirical findings of Chinco et al. (2019). Remarkably, Chinco et al. (2019) apply the LASSO regression to select a small set of predictors from thousands of candidate stocks. They acknowledge that "*the LASSO identifies predictors that are not easy to intuit.*" By Tibshirani's interpretation, their application of the LASSO is implicitly assuming a Laplace prior on the predictive power (i.e., the regression coefficient) of each predictor. This assumption may be challenged for its empirical relevance. Our interpretation can be trouble-free. An econometrician may form a general mixture prior on the predictive power of each predictor in the context of Chinco et al. (2019). If she has little knowledge about the scale of fat-tailed outliers, it can be reasonable to invoke a robust estimate (i.e., the LASSO regression) to shrink most ambiguous estimates to zero.

In our model, traders apply the LASSO estimate to the stock value, not to the stock's predictive power for other stocks. This differs from the application of LASSO in Chinco et al. (2019). We follow the stylized fact that the distribution of stock returns has a sharp peak with fat tails on both sides.[4] It is error-prone to predict extreme events (e.g., Embrechts, Klüppelberg, and Mikosch (2013)). This leads to model risks that can motivate traders to implement robust optimization, voluntarily or mandatorily. Fat tails and model risks are

[3] In our setup, the MAP-based strategy differs from the LASSO strategy whenever the mixture prior is not exactly Laplacian. This heuristic strategy is not an equilibrium outcome and can incur significant losses.

[4] See Fama (1963, 1965), Granger and Ding (1995), and Mantegna and Stanley (1999) for instance.

the two empirically grounded foundations for our theory. By extending our setup to a large number of stocks, we provide an intuitive explanation for the sparse, cross-sectional return predictability documented in Chinco et al. (2019).

Our work attempts to bridge the gap between machine learning (e.g., the LASSO) and neoclassical financial economics. The background model is inspired by the stylized fact of fat tails in asset prices. Unexpected fat-tail shocks can cause temporarily inefficient prices. This feature is embedded in the classic framework of Kyle (1985), which has been extended by many others.[5] By taking the max-min optimization criterion, our model incorporates ambiguity aversion within the framework of Gilboa and Schmeidler (1989).[6]

Our results shed light on an interesting mechanism for limited arbitrage, directly caused by the fat-tailed model risks. This can complement the existing literature which has studied various market issues, such as short-selling costs, leverage constraints, and wealth effects. Those frictions can directly limit arbitrageurs' *ability* to trade; see the survey of Gromb and Vayanos (2010) and the work of Shleifer and Vishny (1997), Xiong (2001), Abreu and Brunnermeier (2002), Gabaix, Krishnamurthy, and Vigneron (2007), Kondor (2009), among others. By excluding those frictions, our model is suited to highlight a mechanism that only affects arbitrageurs' *willingness* to trade. Specifically, it is the prior uncertainty about the fat-tail scale that deters arbitrageurs from eliminating all possible mispricings. Furthermore, their conservative trading, akin to the shrinkage property of LASSO, can mitigate their competition and prevent the asset prices from being fully efficient even when the economy hosts an infinite number of (risk-neutral) arbitrageurs.

Finally, the rational theory in this paper may be applied to various phenomena in behavioral economics, such as the *status quo* bias (Kahneman, Knetsch, and Thaler (1991) and Samuelson and Zeckhauser (1988)) and limited attention (Sims (2003). Our theory may rationalize some algorithmic traders who follow seemingly mechanical trading rules (Lewis (2014)), similar to feedback traders who extrapolate price trends (DeLong, Shleifer, Summers, and Waldmann (1990), Barberis, Greenwood, Jin, and Shleifer (2015, 2018)).

The rest of this paper proceeds as follows. In Section 2, we discuss the background of LASSO. In Section 3, we describe the model setup. In Section 4, we solve the equilibrium. The main results are presented in Section 5, with extensions and applications in Section 6. We make the concluding remarks in Section 7. All proofs are provided in the Appendix.

[5]See Back (1992), Holden and Subrahmanyam (1992), Foster and Viswanathan (1994, 1996), Vayanos (1999, 2001), Back, Cao, and Willard (2000), Yang and Zhu (2020), among others

[6]See Schmeidler (1989), Dow and Werlang (1992), Hansen and Sargent (2001, 2008), Epstein and Schneider (2008, 2010), Easley and O'Hara (2009, 2010), Illeditsch (2011), Banerjee, Davis, and Gondhi (2019). While Klibanoff, Marinacci, and Mukerji (2005) propose a smooth preference model of decisions under ambiguity, we adopt the kinked preference because it is compatible with the experimental results of Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010) and Ahn, Choi, Gale, and Kariv (2014).

2 About the LASSO

We first discuss the definition and the original interpretation of the LASSO method (Tibshirani (1996)). LASSO can simultaneously perform variable selection and regularization. In statistics and machine learning, *regularization* is a technical process that can help simplify the solutions or models, for example, to obtain approximate solutions for ill-posed problems or to prevent overfitting (Hastie et al. (2009)). While a heuristic argument for machine learning can be the principle of parsimony or *Occam's razor*, there is a more formal, statistical perspective which takes many regularization methods as equivalent to imposing certain prior distributions on the model parameters. For example, the LASSO regression features an l_1 regularization, equivalent to imposing a Laplace prior in its objective function. This l_1 penalty is critical for its ability to improve both prediction accuracy and model selection. In contrasts, the *ridge regression* involves an l_2 regularization which is equivalent to imposing a Gaussian prior and thus lacks the ability of variable selection.

In its general form, LASSO is applied to a sample of $I \geq 1$ pairs of predictor-response observations, $\{x_i, y_i\}_{i=1}^I$, where x_i is a vector of $J \geq 1$ covariates (independent variables). After demeaning of the data, the LASSO estimates of $\mathbf{v} = (v_1, ..., v_J)$ are defined by

$$\widehat{\mathbf{v}}^{\text{lasso}} := \arg\min_{\mathbf{v}} \left\{ \frac{1}{2I} \|\mathbf{y} - \mathbf{X} \cdot \mathbf{v}\|_2^2 + \rho \|\mathbf{v}\|_1 \right\}, \tag{1}$$

where $\mathbf{y} = (y_1, ..., y_I)$ is the I-vector of responses (dependent variables), $\mathbf{X}$ is an $I \times J$ covariate matrix, and the positive scalar ρ is the l_1 regularization parameter tuned exogenously. Note that Equation (1) becomes the objective function of the classical OLS regression when $\rho = 0$. Given any positive value of ρ, LASSO forces the sum of the absolute value of regression coefficients, $\|\mathbf{v}\|_1 = \sum_{j=1}^J |v_j|$, to be less than a fixed value. This l_1 penalty forces many insignificant coefficients of $\mathbf{v}$ to zero. It improves the *prediction accuracy* by sacrificing some bias to mitigate the prediction errors. Also, LASSO can enhance the *model interpretability*. It is able to reduce a seemingly high-dimensional prediction problem to a much simpler one, with a sparse subset of coefficients that show the strongest effects.

The simplest version of LASSO corresponds to the setting of $I = J = 1$. Given a single predictor-response pair $\{x_1, y_1\}$, the optimization problem (1) becomes

$$\widehat{v}^{\text{lasso}} := \arg\min_{v} \left\{ \frac{1}{2} |y_1 - x_1 v|^2 + \rho |v| \right\}. \tag{2}$$

The solution to the above problem is given by

$$\widehat{v}^{\text{lasso}} = \begin{cases} (x_1 y_1 - \rho)/x_1^2, & \text{if} \quad x_1 y_1 > \rho, \\ 0, & \text{if} \quad |x_1 y_1| \leq \rho, \\ (x_1 y_1 + \rho)/x_1^2, & \text{if} \quad x_1 y_1 < -\rho. \end{cases} \tag{3}$$

This LASSO solution is closely related to the wavelet shrinkage method developed by Donoho and Johnstone (1994). It is convenient to introduce and define the *soft-thresholding* operator:

$$\mathcal{S}(y; K) = \text{sign}(y) \max(|y| - K, 0) = [y - \text{sign}(y)K]\mathbf{1}_{|y|>K}. \tag{4}$$

Then the LASSO solution (3) can be concisely written as

$$\widehat{v}^{\text{lasso}} = \mathcal{S}(y_1/x_1; \rho/x_1^2) = \mathcal{S}(y_1; \rho/x_1)/x_1, \tag{5}$$

where the estimation threshold is given by ρ/x_1.

Tibshirani proposes that the LASSO can be viewed as the MAP (i.e., posterior mode) estimates when the linear regression coefficients have Laplace (i.e., double-exponential) priors. For the simplest version (2), assume that the prior on v follows a Laplace distribution

$$f_L(v) = \frac{1}{2\xi} \exp\left(-\frac{|v|}{\xi}\right). \tag{6}$$

This density function is sharply peaked since its first derivative is discontinuous at zero. It decays on both sides at the exponential rate ξ and has a raw kurtosis always equal to 6. The likelihood of observing extreme events under a Laplace distribution is much higher than that under the Gaussian distribution with an identical variance. Based on the simple linear regression model, $y_1 = x_1 \tilde{v} + \tilde{u}_1$, where the noise $\tilde{u}_1 \sim \mathcal{N}(0, \sigma_u^2)$ follows the Gaussian distribution, we can apply Bayes' rule to derive the posterior distribution of $\tilde{v}$:

$$f(v|y_1) = \frac{f(y_1|v) f_L(v)}{f(y_1)} = \frac{1}{2\xi f(y_1)\sqrt{2\pi\sigma_u^2}} \exp\left\{-\frac{(y_1 - x_1 v)^2}{2\sigma_u^2} - \frac{|v|}{\xi}\right\}. \tag{7}$$

One can then solve for the MAP estimate under the Laplace prior $\tilde{v} \sim \mathcal{L}(0, \xi)$,

$$\begin{aligned} \widehat{v}^{\text{map},\mathcal{L}} &= \arg\max_v f(v|y_1) = \arg\min_v \left\{ \frac{(y_1 - x_1 v)^2}{2\sigma_u^2} + \frac{|v|}{\xi} \right\} && (8) \\ &= \frac{1}{x_1} \mathcal{S}(y_1; \sigma_u^2/(x_1\xi)) = \widehat{v}^{\text{lasso}}. && (9) \end{aligned}$$

This coincides with the LASSO estimate (3) if we assign $\rho = \sigma_u^2/\xi$. The Laplace distribution concentrates its probability mass around zero than does the normal distribution. The MAP method combined with such a sharply peaked prior tends to set a range of estimates to zero, operationally equivalent to the role of the l_1 norm regularization in Eq. (2).

Note that the original mathematical definition of LASSO in its Lagrangian form does not require the specification of a prior distribution on v. Only the later interpretation of LASSO may need us to specify the prior. For this reason, there could be different interpretations in principal. We restate the two problems with the statistical interpretation of LASSO.

First, for its use of the posterior mode, the MAP estimation is not Bayesian rational. When the posterior distribution is skewed, the mode usually differs from the mean and causes estimation bias. A Bayesian rational agent should follow the posterior mean estimate which integrates all the relevant information. The mode estimate can ignore useful information and has to be viewed as a heuristic method (due to bounded nationality).

Second, the MAP-based argument for LASSO only works with a pure fixed Laplace prior. If we replace it with a slightly different prior or deal with any uncertainty about this prior, the MAP argument will not produce the LASSO objective. In reality, it seems too restrictive to impose a pure and fixed Laplace prior on the model parameter. One might defend that this prior is just an approximation or a simple device to model sparsity. The point is that the general definition (1) of LASSO does not ask us to specify such a prior. The LASSO itself is invented as a tool to solve sparse learning problems, not a model to describe them.

Based on these discussions, the statistical interpretation of LASSO is both heuristic (in learning) and restrictive (in prior). This does not mean that the LASSO itself is heuristic or restrictive. A major point made in this paper is that the LASSO method admits a Bayesian rational interpretation which is applicable to flexible, uncertain priors.

To formalize our intuition, we develop a simple model of arbitrage trading which incorporates the ingredients of fat tails, model risks, and robust control. In our setup, traders follow the rational Bayesian learning rule to evaluate all possible states and they obey sequential rationality to optimize the standardized max-min expected utilities. In contrast to a pure fixed Laplace prior which has a raw kurtosis of 6, agents in our model have an uncertain fat-tail prior described by the Gaussian-Laplacian mixture distribution which has a kurtosis ranging from 3 to 6.125. When agents are uncertain about the scale of fat-tail shocks, we show that their max-min robust strategy is exactly a LASSO algorithm in equilibrium.

3 Model

Arbitrage opportunities are often short-lived and unexpected by the general market. We develop a model to analyze how arbitrageurs may capitalize on such opportunities. As the model's background, we first need a trading environment that occasionally produces pricing errors. This is achieved in a two-period setup by incorporating random fat-tail shocks to disrupt a presumably efficient market. We then introduce arbitrageurs to exploit mispricings.

Our model is based on three key assumptions: (1) the general Laplacian-Gaussian mixture distribution for the asset's liquidation value; (2) the linear price adjustments in responses to total order flows, which contain informed trading proportional to the residual information; (3) the strategic arbitrageurs who secretly exploit pricing errors by optimizing their max-min expected utilities under uncertain fat-tail priors (as model risk). The linear pricing schedules and the non-Gaussian asset values imply the occurrences of pricing errors. Our focus is to study how arbitrageurs act in this uncertain fat-tailed environment.

Fat Tails. Consider a market with a single risky asset and two rounds of trading, indexed by $t = 1, 2$. The asset liquidation value follows a Laplacian-Gaussian mixture distribution, which is denoted as $\tilde{v} \sim \mathcal{LG}(\alpha, \xi_v)$ and described by the probability density function,

$$f(v) = \frac{\alpha}{2\xi_v} \exp\left(-\frac{|v|}{\xi_v}\right) + \frac{1-\alpha}{\sqrt{2\pi\sigma_v^2}} \exp\left(-\frac{v^2}{2\sigma_v^2}\right). \tag{10}$$

It depends on the mixing weight α (the frequency of fat-tail shocks), the fat-tail scale parameter ξ_v (the dispersion of fat-tail shocks), and the Gaussian variance σ_v^2. In simulations, $\tilde{v}$ is randomly drawn from either a Gaussian or a Laplacian distribution, since we can write

$$\tilde{v} = (1-\tilde{s}) \cdot \tilde{v}_G + \tilde{s} \cdot \tilde{v}_L, \quad \text{with} \quad \tilde{v}_G \sim \mathcal{N}(0, \sigma_v^2) \quad \text{and} \quad \tilde{v}_L \sim \mathcal{L}(0, \xi_v), \tag{11}$$

where $\tilde{s}$ is a Bernoulli random variable that is equal to 1 with probability α and to 0 with probability $1-\alpha$. In Eq. (11), the distributional type of $\tilde{v}$ is encoded by the value of $\tilde{s}$.

Empirically, the distribution of stock returns exhibits one sharp peak and two fat tails. It can be reasonably characterized by the above mixture distribution (Lillo and Mantegna (2000), Silva et al. (2004), Haas, Mittnik, and Paolella (2006), Behr and Pötter (2009)). The picture is also consistent with the stylized fact that the stock market experiences jumps.

Theoretically, the mixture prior combines two well-known distributions which are simple and stable.[7] It may be viewed as a microstructure snapshot of the jump-diffusion process

[7] As discussed by Fama (1963, 1965) and Rachev and SenGupta (1993), it is appealing to model stock returns as realizations of some stable distribution. The Gaussian distribution is Lévy stable, whereas the Laplace distribution is geometric stable. See Kotz, Kozubowski, and Podgórski (2001).

assumed in the option pricing model of Kou (2002). The fat-tail component ($\tilde{v}_L$) in Eq. (11) can create sparse arbitrages if the general market believes that both informed and noise demands follow Gaussian distributions (e.g., Kyle (1985)). The general structure of Eq. (10) covers the Laplace distribution as a special case (i.e., $\alpha = 1$). This is convenient when we refer to the statistical interpretation of LASSO since it hinges on a pure Laplace prior.

Linear Pricing. Price movements are assumed to be linear in the total order flows $\tilde{y}_t$:

$$\tilde{p}_1 - p_0 = \lambda_1 \tilde{y}_1, \qquad \tilde{p}_2 - \tilde{p}_1 = \lambda_2 \tilde{y}_2, \tag{12}$$

where $\lambda_t > 0$ is the price impact per unit of order flow at time t. We can set the initial price $p_0 = 0$ without loss of generality. The total order flow $\tilde{y}_t$ contains private information which, by dynamical consistency, is proportional to the residual information $\tilde{v} - \tilde{p}_{t-1}$. This is a common feature of dynamic trading models following Kyle (1985). To avoid the information from being fully revealed, the order flows are contaminated by noise trading demands, which obey the Gaussian law, $\tilde{u}_1 \sim \mathcal{N}(0, \sigma_u^2)$ and $\tilde{u}_2 \sim \mathcal{N}(0, \gamma\sigma_u^2)$, with time-varying volatilities tuned by the parameter $\gamma > 0$. All the random variables $\tilde{v}$, $\tilde{u}_1$, and $\tilde{u}_2$ are mutually independent. Thus, before considering the arbitrage trading, the total order flows are

$$\tilde{y}_1 = \beta_1(\tilde{v} - p_0) + \tilde{u}_1, \qquad \tilde{y}_2 = \beta_2(\tilde{v} - \tilde{p}_1) + \tilde{u}_2, \tag{13}$$

where $\beta_t > 0$ is the trading intensity on the remaining information $\tilde{\theta}_t := \tilde{v} - \tilde{p}_{t-1}$ at time t. Eq. (12) and Eq. (13) together describe the trading environment of our model. This can be embedded in a linear subgame-perfect Markov equilibrium of a classical trading model, such as Kyle (1985), Holden and Subrahmanyam (1992), Foster and Viswanathan (1996). An example (microfoundation) is presented below, with details discussed in Appendix A.1.

Example. Consider the two-period case of the multi-period model in Holden and Subrahmanyam (1992), where $M \geq 1$ informed traders privately observe the value of $\tilde{v}$ at $t = 0$. Market makers believe that $\tilde{v}$ is drawn from the Gaussian distribution $\mathcal{N}(0, \sigma_v^2)$. There exists a unique linear equilibrium where the asset price moves linearly as in Eq. (12) with

$$\lambda_1 = \frac{\sqrt{M(M+1)^2[(M+1)^2 - 2/\delta]}}{(M+1)^3 - 2M/\delta} \cdot \frac{\sigma_v}{\sigma_u}, \qquad \lambda_2 = \delta\lambda_1 = \sqrt{\frac{\delta M/\gamma}{\delta(M+1)^3 - 2M}} \cdot \frac{\sigma_v}{\sigma_u}. \tag{14}$$

Here, the ratio $\delta := \lambda_2/\lambda_1$ is the root of the cubic equation, $(M+1)^4\gamma\delta^3 - 2(M+1)^2\gamma\delta^2 - (M+1)^3\delta + 2M = 0$, subject to the constraint $\delta(M+1)^2 > 2$ and the second-order conditions. The aggregate order flows in this example obey the form of Eq. (13). The intensities of aggregate informed trading in Eq. (13) are found to be

$$\beta_1 = \frac{\delta M(M+1)^2 - 2M}{\lambda_1[\delta(M+1)^3 - 2M]}, \qquad \beta_2 = \frac{M}{\lambda_2(M+1)}. \tag{15}$$

When $M = 1$, Eq. (14) and Eq. (15) reproduce the solution for the two-period Kyle model.

In general, the linear pricing rule (12) is efficient only when $\tilde{v}$ is Gaussian (i.e., $\alpha = 0$). If $\tilde{v}$ actually follows the mixture distribution (10) with $\alpha \in (0, 1]$ and if its fat-tailed part is unexpected by the market, then pricing errors will take place with probability α. The generic results in this paper do not depend on the content of λ_t or β_t. For this reason, both λ_t and β_t will be treated as exogenously given and commonly known. The above example provides a microfoundation which is not unique but general enough for numerical purposes.

The assumption of linear price changes (12) has both theoretical and empirical relevance. Huberman and Stanzl (2004) show that if the price impact of trades is both permanent and time-independent, then only linear price impact functions can rule out quasi-arbitrage and support viable market prices. Empirically, the price impact function is found to be sublinear for small and medium-size orders, with moderate concavity for large orders.[8]

The linear pricing rule implies that market makers may have Gaussian beliefs or other concerns (e.g., robustness) that effectively restrict themselves to the linear pricing strategy. It is perhaps the case that the market has ignored some sparse, short-lived arbitrages (Chinco et al. (2019)) or that some information is too subtle to be noticed by average traders (Deng, Gao, Hu, and Zhou (2020)). Price inefficiency caused by incorrect beliefs may not persist in the long term because the market may gradually learn and fix the problem. If we follow the standard assumption in Kyle-type models that market makers know the true distributions and set prices as efficient as possible, then their optimal pricing strategy should be nonlinear and convex given the fat tails of $\tilde{v}$. This is at odds with the empirical price impact function. Another possibility is that the objective of market makers in reality may be different from what has been assumed in standard models. When the trading environment receives various uncertain shocks, market makers may become concerned more about the robustness than about the efficiency of their pricing schedules. The linear pricing rule is simple and robust, allowing them to readily tune the price impact parameters λ_t and avoid losses on average. This argument can sustain inefficient prices, The linear pricing rule can lose efficiency when the asset value is fat-tailed: it tends to underestimate the information content in large orders. The frequency and the magnitude of mispricings are controlled by α and ξ_v, respectively. This is how our setup generates arbitrage opportunities and opens the door to arbitrageurs.

[8] See Loeb (1983), Lillo and Mantegna (2000), Grinold and Kahn (2000) [p. 453], Hasbrouck and Seppi (2001), Plerou, Gopikrishnan, Gabaix, and Stanley (2002), Gabaix et al. (2006), Kyle and Obizhaeva (2016).

Cautious Arbitrageurs. Consider a number of arbitrageurs, indexed by $n = 1, ..., N$. They are sophisticated enough to know about the distributional structure of $\tilde{v}$. They do not observe $\tilde{v}$ until it is revealed to the public at $t = 3$. Each arbitrageur can secretly place two orders, $\tilde{z}_{1,n}$ and $\tilde{z}_{2,n}$, to exploit the short-term mispricings. Their strategy profile is denoted by a matrix of real-valued functions, $\mathbf{Z} = [\mathbf{Z}_1, ..., \mathbf{Z}_N]$ where $\mathbf{Z}_n = \langle Z_{1,n}, Z_{2,n} \rangle$ is the n-th arbitrageur's strategy for $n = 1, ..., N$. The quantities traded by the n-th arbitrageur are $\tilde{z}_{1,n} = Z_{1,n}(p_0)$ and $\tilde{z}_{2,n} = Z_{2,n}(\tilde{p}_1)$. The trading profit for the n-th trader is $\tilde{\pi}_{z,n} := \sum_{t=1}^{2} (\tilde{v} - \tilde{p}_t)\tilde{z}_{t,n}$. Once we take into account their trading activities, the actual order flow at time $t \in \{1, 2\}$ is

$$\tilde{y}_t = \beta_t(\tilde{v} - \tilde{p}_{t-1}) + \sum\nolimits_{n=1}^{N} \tilde{z}_{t,n}(\tilde{p}_{t-1}) + \tilde{u}_t. \tag{16}$$

Arbitrageurs are regarded as strategic institutional traders who know the total number (N) of competitors. In our model, they could infer α from the sample kurtosis of realized stock values, although our key results remain when traders are uncertain about α.[9] Considering the difficulty in predicting jump events (Bollerslev and Todorov (2011a,b)), we assume that arbitrageurs do not know exactly when the pricing errors would take place (i.e., they do not observe $\tilde{s}$ in Eq. (11)) or how severe the errors would be (i.e., they are uncertain about ξ_v).

To formalize the model risk faced by arbitrageurs, we express their uncertain priors as $\tilde{v} \sim \mathcal{LG}(\alpha, \tilde{\xi})$, where $\tilde{\xi} \in [\xi_L, \xi_H]$ can take any value between the lowest and highest priors. With this fat-tailed model risk, arbitrageurs may care about the robustness of their strategies. Gilboa and Schmeidler (1989) axiomatize the *max-min expected utility* theory as a standard, rational framework for modeling ambiguity-averse preferences. We follow this classic theory by assuming that each arbitrageur's objective is to maximize the minimum expected trading profit over all possible priors. Each trader will internalize the price impacts of all traders.

Definition of Equilibrium. Given the price function (12) and the aggregate order flows (16), we define a sequential trading (partial) equilibrium among arbitrageurs who have uncertain fat-tail priors $\mathcal{LG}(\alpha, \tilde{\xi})$ with $\tilde{\xi} \in [\xi_L, \xi_H]$. The equilibrium is described by a matrix of their strategies $\mathbf{Z}$ such that for all $n = 1, ..., N$ and any alternative strategy profile $\mathbf{Z}'$ that differs from $\mathbf{Z}$ only in the n-th entry $\mathbf{Z}'_n = \langle Z'_{1,n}, Z'_{2,n} \rangle$, the strategy profile $\mathbf{Z}$ yields a utility level (i.e., the minimum expected profit over all possible priors) no less than $\mathbf{Z}'$, and $Z_{2,n}$ yields a utility level in the second period no less than that produced by any single deviation $Z'_{2,n}$:

$$\min_{\xi} \mathrm{E}^{\mathcal{A}}[\tilde{\pi}_{z,n}(\mathbf{Z}) | \xi_v = \xi] \quad \geq \quad \min_{\xi} \mathrm{E}^{\mathcal{A}}[\tilde{\pi}_{z,n}(\mathbf{Z}') | \xi_v = \xi], \tag{17}$$

$$\min_{\xi} \mathrm{E}^{\mathcal{A}}[(\tilde{v} - \tilde{p}_2(\cdot, Z_{2,n}))Z_{2,n} | \tilde{p}_1, \xi_v = \xi] \quad \geq \quad \min_{\xi} \mathrm{E}^{\mathcal{A}}[(\tilde{v} - \tilde{p}_2(\cdot, Z'_{2,n}))Z'_{2,n} | \tilde{p}_1, \xi_v = \xi]. \tag{18}$$

[9] Given the mixture distribution (10), the kurtosis of $\tilde{v}$ is $3 + 3(4+\alpha)(1-\alpha)/(2-\alpha)^2$ which only depends on α; see Haas et al. (2006). Section 6.1 discusses the case when traders are uncertain about both α and ξ_v.

4 Equilibrium Strategies

4.1 Optimal strategy under a fixed prior

Arbitrageurs' trading strategies are driven by their estimates of the extent to which the asset has been mispriced. Conditional on past prices, arbitrageurs' expectations of $\tilde{v}$ depend on their fat-tail priors $\mathcal{LG}(\alpha, \tilde{\xi})$ with $\alpha \in (0, 1]$. When there is no model risk about ξ at all, arbitrageurs become the standard *subjective expected utility* optimizers, under a fixed fat-tail prior $\mathcal{LG}(\alpha, \xi)$. This leads to the benchmark trading strategy in this paper.

Theorem 1. *Suppose arbitrageurs have the same fixed Laplacian-Gaussian prior $\mathcal{LG}(\alpha, \xi)$ where $\alpha \in [0, 1]$ and $\xi \in (0, \infty)$. There exists a symmetric equilibrium where they choose to watch the market without any betting at $t = 1$, i.e., $Z^o_{1,n} = 0$, and their optimal strategy at $t = 2$ is proportional to their posterior mean estimate $\widehat{\theta}$ of the pricing error $\tilde{\theta} := \tilde{v} - p_1$, i.e.,*

$$Z^o_{2,n}(p_1; \alpha, \xi) = \frac{1 - \beta_2\lambda_2}{\lambda_2(N+1)} \cdot \widehat{\theta}(p_1; \alpha, \xi) = \frac{1 - \beta_2\lambda_2}{\lambda_2(N+1)} \cdot [\widehat{v}(p_1; \alpha, \xi) - p_1]. \tag{19}$$

Here, $\widehat{v} := \mathrm{E}^{\mathcal{A}}[\tilde{v}|p_1, p_0]$ is the posterior mean estimate of $\tilde{v}$ under arbitrageurs' prior $\mathcal{LG}(\alpha, \xi)$ and conditional on the price history. Given the linear pricing rule (12), $\widehat{v}$ only depends on the order flow y_1 which drives the price change $p_1 - p_0$. If we measure order flows in units of the noise volatility, $y := y_1/\sigma_u$, and define $\kappa := \sigma_u/(\beta_1\xi)$, then the posterior mean of $\tilde{v}$ is

$$\widehat{v}(y) = \frac{\alpha\sigma_u(y-\kappa)\mathrm{erfc}\left(\frac{\kappa-y}{\sqrt{2}}\right)}{\beta_1\mathrm{erfc}\left(\frac{\kappa-y}{\sqrt{2}}\right) + \beta_1 e^{2\kappa y}\mathrm{erfc}\left(\frac{\kappa+y}{\sqrt{2}}\right)} + \frac{\alpha\sigma_u(y+\kappa)\mathrm{erfc}\left(\frac{\kappa+y}{\sqrt{2}}\right)}{\beta_1\mathrm{erfc}\left(\frac{\kappa+y}{\sqrt{2}}\right) + \beta_1 e^{-2\kappa y}\mathrm{erfc}\left(\frac{\kappa-y}{\sqrt{2}}\right)} + (1-\alpha)\lambda_1\sigma_u y. \tag{20}$$

This function strictly increases with the order flow and has two shape parameters α and $\kappa(\xi)$. Asymptotically, $\widehat{v}$ becomes linear in the total order flow realized at $t = 1$:

$$\widehat{v} \to \alpha[y_1 - \mathrm{sign}(y_1)\kappa\sigma_u]/\beta_1 + (1-\alpha)\lambda_1 y_1, \quad \text{as} \quad |y_1| \to \infty. \tag{21}$$

Proof. See Appendix A.2. □

Given Eq. (20) and $p_1 = \lambda_1 y_1$, one can factor out α from the strategy function (19):

$$Z^o_{2,n}(y_1; \alpha, \xi) = \alpha Z^o_{2,n}(y_1; \alpha = 1, \xi) = \frac{\alpha(1 - \beta_2\lambda_2)[\widehat{v}(y_1; \alpha = 1, \xi) - \lambda_1 y_1]}{\lambda_2(N+1)}. \tag{22}$$

The trading intensity is exactly proportional to the frequency of fat-tail shocks. Therefore, the key mathematical properties of $Z^o_{2,n}(y_1;\alpha,\xi)$ is independent of α. This scaling rule is an important feature of all Bayesian rational strategies discussed in this paper.

Because arbitrageurs' prior $\mathcal{LG}(\alpha,\xi)$ is a symmetric distribution, they tend to postpone trading until they can distinguish the direction of signals (as the posterior becomes skewed). When solving the equilibrium, we conjecture first and verify later that arbitrageurs do not trade at $t=1$. This is confirmed by Theorem 1 and explains our choice of a two-period setup. The no-trade conjecture holds if the market is not too crowded for arbitrageurs; otherwise, it can be profitable for a trader to trade at $t=1$ in the hope that other traders get misled.

Proposition 1. *For $\xi>0$ and $\alpha\in[0,1]$, the symmetric equilibrium in Theorem 1 exists if the following market condition holds:*

$$1+\frac{\alpha(1-\beta_1\lambda_1)}{\beta_1\lambda_1}\cdot\frac{N-1}{N+1}<\frac{2\sqrt{\lambda_2/\lambda_1}}{1-\beta_2\lambda_2}. \tag{23}$$

Proof. See Appendix A.3. □

The condition (23) turns out to be a universal condition for all the equilibria analyzed in this paper. Unless otherwise specified, this (sufficient) condition is assumed to hold hereafter. This precludes unilateral deviations from the no-trade strategy at $t=1$. According to (23), the equilibrium can accommodate an infinite number of arbitrageurs when the trading environment satisfies: $\alpha+(1-\alpha)\beta_1\lambda_1<2\beta_1\sqrt{\lambda_1\lambda_2}/(1-\beta_2\lambda_2)$.[10]

Corollary 4.1. *The strategy function $Z^o_{2,n}(\tilde{p}_1;\alpha,\xi)=Z^o_{2,n}(\tilde{y}_1;\alpha,\xi)$ defined by Eq. (19) and Eq. (20) has the following properties:*
(a) It is a smooth, odd function of y_1 and symmetric about the origin.
(b) It is a convex function for $y_1\geq 0$ and a concave function for $y_1\leq 0$.
(c) It has two slant asymptotes for any $\xi\in(0,\infty)$ which take the general form below,

$$Z^\infty(y_1;\alpha,\xi)=\frac{\alpha(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)}{\beta_1\lambda_2}\cdot\frac{y_1-\operatorname{sign}(y_1)K(\xi)}{N+1}, \tag{24}$$

where the horizontal intercept $K(\xi)$ is inversely related to ξ but independent of α,

$$K(\xi)=\frac{\kappa(\xi)\sigma_u}{1-\beta_1\lambda_1}=\frac{\sigma_u^2\xi^{-1}}{\beta_1(1-\beta_1\lambda_1)}. \tag{25}$$

[10] For example, if λ_t and β_t are determined by the two-period Kyle model (see Proposition 1 of Huddart, Hughes, and Levine (2001)), one can verify that the equilibrium condition (23) holds even when $N\to\infty$.

(d) There exists a unique critical value, $\xi_c > 0$, endogenously determined by the equation:

$$1 + \left(\frac{\sigma_u}{\beta_1 \xi_c}\right)^2 - \sqrt{\frac{2}{\pi}} \frac{\sigma_u}{\beta_1 \xi_c} \frac{\exp(-\sigma_u^2/(2\beta_1^2 \xi_c^2))}{\operatorname{erfc}(\sigma_u/(\sqrt{2}\beta_1 \xi_c))} = \beta_1 \lambda_1. \tag{26}$$

For $\xi \geq \xi_c$, $Z^o_{2,n}(y_1; \alpha, \xi)$ is an increasing function of y_1 and it has only one root at $y_1 = 0$. For $\xi < \xi_c$, $Z^o_{2,n}(y_1; \alpha, \xi)$ is a non-monotonic function of y_1 and it has three different roots. Note that ξ_c is independent of α and ξ_v; it depends on the ratio σ_u/β_1 and the product $\beta_1 \lambda_1$.

Proof. See Appendix A.4. □

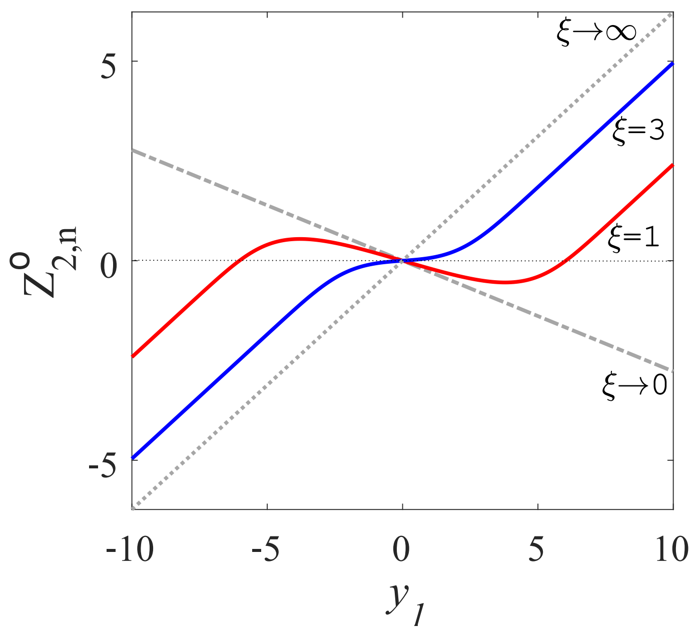

Figure 1. The subjective optimal strategy $Z^o_{2,n}(y_1; \alpha, \xi)$ in Eq. (19) for various values of ξ.

The main properties listed in Corollary 4.1 are reflected in Figure 1 where we plot the optimal strategy (19) for different values of ξ. An arbitrageur with the extreme prior $\xi \to 0$ believes that the asset value is unchanged (i.e., $\tilde{v} = 0$). This trader will attribute all the order flow y_1 to noise trading and trade against any price movements. In contrast, an arbitrageur with the extreme prior $\xi \to \infty$ believes that the first-period order flow was dominated by informed trading and will chase the price trend straightly. When $\xi < \xi_c$, this strategy is contrarian ("leaning against the wind") for small order flows but momentum for sufficiently large ones. Fatter tails in the prior (i.e., larger ξ) lead to a lower threshold for switching to momentum trading. When $\xi \geq \xi_c$, the strategy is always trend-following.

4.2 Robust strategy under uncertain priors

Model risks can be prominent in a fat-tailed trading environment where market meltdown may be triggered if some big trader has applied a wrong model. Institutional traders are often required to test their strategies across alternative scenarios. This pressure can motivate them to adopt strategies that sacrifice some optimality for robustness.

In our setup, how would arbitrageurs trade given their uncertain fat-tail prior $\mathcal{LG}(\alpha, \tilde{\xi})$? Figure 1 shows that they will face ambiguity about the profitable trading direction when they observe small and medium order flows. They may want to buy this asset under a high prior (blue line) but sell it under a low prior (red line). In case they use the wrong prior, they may trade on the wrong side and expose themselves to adverse fat-tail shocks. For robustness, they should not trade until there is little ambiguity about the trading direction.

Corollary 4.2. *When the range of uncertain Laplace prior satisfies $\xi_L < \xi_c \leq \xi_H$,[11] there is an equilibrium where arbitrageurs idle at $t = 1$ and their pure max-min strategy at $t = 2$ is $Z^o_{2,n}(y_1; \alpha, \xi_L)\mathbf{1}_{|y_1|>K_L}$, where $K_L(\xi_L)$ is the positive root of the equation: $Z^o_{2,n}(y_1; \alpha, \xi_L) = 0$. The solution of K_L is independent of α, according to the scaling property (22).*

Proof. See Appendix A.5. ☐

The trading strategy $Z^o_{2,n}(y_1; \alpha, \xi_L)\mathbf{1}_{|y_1|>K_L}$ in Corollary 4.2 is extremely biased because it is determined by the lowest prior ξ_L and its trading threshold K_L can be arbitrarily large when ξ_L is arbitrarily small. This strategy may sacrifice too much optimality for robustness, making the equilibrium in Corollary 4.2 undesirable in reality.

Arbitrageurs may ponder a different equilibrium where they are not attached to the lowest prior ξ_L. This *debiased* equilibrium may be more attractive as it may balance robustness and optimality. In any possible equilibrium, the asymptotes of arbitrageurs' strategies at $t = 2$ cannot slope differently from Eq. (24); otherwise, for sufficiently large y_1, they would trade either more than the most optimistic strategy (following ξ_H) or less than the most pessimistic strategy (following ξ_L). Thus, in a symmetric and debiased equilibrium, arbitrageurs' strategies must have the same asymptotes in the form of (24), denoted $Z^\infty(y_1; \alpha, \xi_w)$ for some $\xi_w > \xi_L$. Arbitrageurs can average across multiple priors to achieve $\xi_w > \xi_L$. There is a formal argument for this debiasing mechanism: observing a sufficiently large order flow y_1 may convince arbitrageurs that y_1 contains a strong fat-tail signal, which may ease their concerns about trading on the wrong side and make them indifferent to model risks. Suppose they become ambiguity-neutral as $y_1 \to \pm\infty$. Then the asymptotes of their strategies must coincide

[11] If $\xi_c \leq \xi_L < \xi_H$, the max-min strategy is $Z^o_{2,n}(y_1; \alpha, \xi_L)$ but it means $\xi_v < \xi_L < \xi_H$. If $\xi_L < \xi_H < \xi_c$, the max-min strategy is over-complicated and lacks empirical relevance. See Figure 11 and Appendix A.5.

with the asymptotes of their risk-neutral strategies which, after averaging across all possible priors, is given by $\mathrm{E}[Z^o_{2,n}(y_1;\alpha,\tilde{\xi})]$. For an arbitrary weight function $w\colon[\xi_L,\xi_H]\to[0,1)$ with $\int_{\xi_L}^{\xi_H} w(\xi)d\xi=1$, Eq. (24) and Eq. (25) imply that ξ_w is the weighted harmonic mean of $\tilde{\xi}$:

$$\xi_w := \left(\int_{\xi_L}^{\xi_H} \xi^{-1} w(\xi)d\xi\right)^{-1}. \tag{27}$$

Depending on the weight function $w(\xi)$, the weighted average ξ_w can take any value over the interval (ξ_L,ξ_H) and correspondingly, the asymptotes $Z^\infty(y_1;\alpha,\xi_w)$ can shift freely as well.

Theorem 2. *There exists a symmetric, debiased equilibrium if the following conditions hold:*

(C1) arbitrageurs' admissible strategies converge to $Z^\infty(y_1;\alpha,\xi_w)$ with $K(\xi_w)<K_L$;

(C2) arbitrageurs' admissible strategies are convex for $y_1\geq 0$ and concave for $y_1\leq 0$;

(C3) the range of their uncertain priors satisfies $\xi_L<\xi_c\leq\xi_H$, where ξ_c solves Eq. (26).

In this equilibrium, arbitrageurs watch the market without any trading at $t=1$. Their robust trading strategy at $t=2$ is a soft-thresholding function of the order flow y_1 realized at $t=1$:

$$\begin{aligned} Z_{2,n}(y_1;\alpha,K(\xi_w)) &= Z^\infty(y_1;\alpha,\xi_w)\mathbf{1}_{|y_1|>K(\xi_w)} = \frac{\alpha(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)}{\beta_1\lambda_2(N+1)}\mathcal{S}(y_1;K(\xi_w)) \\ &= \frac{\alpha(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)}{\beta_1\lambda_2(N+1)}\left[y_1-\mathrm{sign}(y_1)K(\xi_w)\right]\mathbf{1}_{|y_1|>K(\xi_w)}. \end{aligned} \tag{28}$$

The equilibrium existence condition is still given by the inequality (23).

Proof. See Appendix A.6. □

As discussed earlier, the first condition (C1) is endogenously implied by the existence of a symmetric and debiased equilibrium. (C1) can be microfounded, for example, by assuming arbitrageurs are asymptotically indifferent to the model risk. If (C1) is absent, the condition (C2) alone will have no effect because this model economy will admit the same equilibrium stated in Corollary 4.2 which only requires the condition (C3). Only when (C1) holds, the condition (C2) can play a meaningful role in *regularizing* the max-min problem under (C1).[12] Note that (C2) is consistent with the curvature property of optimal strategies (Corollary 4.1(b) and Figure 1). In practice, (C2) can be a desirable property to prevent overfitting.

Given (C1) and (C2), the admissible strategies must be enclosed by $Z^o_{2,n}(y_1;\alpha,\xi\to\infty)$, $Z^o_{2,n}(y_1;\alpha,\xi\to 0)$, and $Z^\infty(y_1;\alpha,\xi_w)$; see the shaded area in Figure 2. Any strategy that

[12] Without the curvature condition (C2), for any strategy that satisfies (C1), arbitrageurs can always find some deviations that trade more conservatively than this strategy. Such deviations are preferred under the max-min criterion but fail to support an equilibrium. Note that the deviations are possble because of the gap between $Z^\infty(y_1;\alpha,\xi_w)$ and $Z^\infty(y_1;\alpha,\xi_L)$, which is a direct implication of (C1).

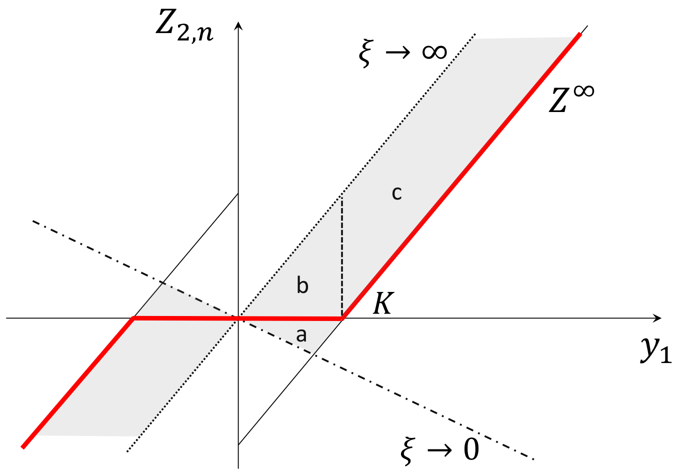

Figure 2. The robust trading strategy $Z_{2,n}(y_1; \alpha, K)$ under model risk as in Theorem 2.

goes out of the shaded area is either irrational or violating (C1) or (C2). We focus on the positive domain of y_1 and divide the shaded area into three regions. First, for $y_1 \in [0, K(\xi_w)]$, each arbitrageur will not sell against y_1, considering that she may lose money on average by doing so if the highest prior ξ_H is true. This rules out any decision point inside the triangle "a". Similarly, each arbitrageur will not purchase this asset, considering that she may lose money if the lowest prior ξ_L is true. This rules out any decision point inside the triangle "b". By the max-min criterion, each arbitrageur will not trade for $y_1 \in [0, K(\xi_w)]$. Next, for any $y_1 \in (K(\xi_w), \infty)$, each arbitrageur will not trade more than the amount of $Z^\infty(y_1; \alpha, \xi_w)$, because she understands that in the worst case she could lose more money by trading more. This argument rules out any decision point inside the open region "c". By symmetry, their equilibrium strategy is the red line in Figure 2, which is exactly characterized by Eq. (28).

This robust strategy $Z_{2,n}(y_1; \alpha, K(\xi_w))$ is a soft-thresholding function of the total order y_1. Its slope $\frac{\alpha(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)}{\beta_1\lambda_2(N+1)}$ is independent of ξ_w, whereas its threshold $K(\xi_w) = \frac{\sigma_u^2 \xi_w^{-1}}{\beta_1(1-\beta_1\lambda_1)}$ is independent of α. The no-trade zone $[-K(\xi_w), K(\xi_w)]$ indicates infrequent trading activities at $t = 2$. Because arbitrageurs do not trade at $t = 1$ and only trade occasionally at $t = 2$, a range of small pricing errors can survive in this market. *Ex post*, an econometrician may find pervasive anomalies after analyzing the trading data in this model economy. She may question the rationality or capability of arbitrageurs. *Ex ante*, arbitrageurs have rationally assessed all the possible states and they let go many vague mispricings for robustness. There are no exogenous frictions that limit their trading ability, except the price impact costs, λ_1 and λ_2, which can be endogenized in a standard Kyle-type model. The major friction in our setup is the fat-tailed model risk which discourages cautious arbitrageurs from eradicating all possible pricing errors.

5 Main Results

5.1 LASSO as a Bayesian rational strategy

A strategy is a complete plan of action a player will take contingent on what circumstances might arise. The choice of strategy usually depends on how the player predicts or estimates relevant variables. When the prior is known and fixed, a Bayesian rational player will use the posterior mean estimate to maximize her utility. When the prior is unknown or uncertain, the player may take one of two routes: either sacrifice rationality to alter her learning method or maintain rationality by optimizing a different utility function. When both routes lead to the same strategy, we cannot distinguish which route the player has actually taken.

Given the standard definition of LASSO estimates (Eq. (1) in Section 2), we now define the LASSO strategy as a plan of action which can be exactly and effectively implemented by using the LASSO estimate(s) of economic variable(s). This definition only requires *observational equivalence*. A player's motivation of using some machine learning technique is not directly observable. If a player chooses a strategy indistinguishable from the strategy that directly applies the XYZ technique, then we can only call it an XYZ strategy in the sense of implementation. The definition itself does not take any stance on the player's motivation in developing the XYZ strategy. To rationalize the use of XYZ technique, we need a system of economic arguments to show that some Bayesian rational strategy is an XYZ strategy. It is this system of arguments that constitutes the economic rationale for using the technique.

Theorem 3. *For any $\alpha \in (0, 1]$, the robust strategy (28) in Theorem 2 is a LASSO strategy:*

$$Z_{2,n}(y_1; \alpha, K(\xi_w)) = \frac{\alpha(1-\beta_2\lambda_2)(\widehat{v}^{\text{lasso}} - \lambda_1 y_1)\mathbf{1}_{|y_1|>K(\xi_w)}}{\lambda_2(N+1)} = \frac{\alpha(1-\beta_2\lambda_2)}{\lambda_2(N+1)} \cdot \widehat{\theta}^{\text{lasso}}. \tag{29}$$

where the LASSO estimate $\widehat{v}^{\text{lasso}}$ is the solution to the LASSO objective function:

$$\widehat{v}^{\text{lasso}}(y_1; \xi_w) := \arg\min_v \left\{ \frac{1}{2}|y_1 - \beta_1 v|^2 + \frac{\sigma_u^2}{\xi_w}|v| \right\} = \frac{1}{\beta_1}\mathcal{S}(y_1; \kappa\sigma_u). \tag{30}$$

and the LASSO estimate $\widehat{\theta}^{\text{lasso}}$ of the mispricing $\tilde{\theta} = \tilde{v} - p_1$ solves the LASSO objective:

$$\widehat{\theta}^{\text{lasso}}(y_1; \xi_w) := \arg\min_\theta \left\{ \frac{1}{2}\left| y_1 - \frac{\beta_1\theta}{1-\beta_1\lambda_1}\right|^2 + \frac{\sigma_u^2|\theta|}{(1-\beta_1\lambda_1)^2\xi_w} \right\} = \frac{1-\beta_1\lambda_1}{\beta_1}\mathcal{S}(y_1; K(\xi_w)). \tag{31}$$

The two thresholds satisfy $\kappa_a\sigma_u < K(\xi_w) = \kappa_a\sigma_u/(1-\beta_1\lambda_1)$ since $\beta_1\lambda_1 \in (0, 1)$.

Proof. See Appendix A.7. □

Theorem 3 shows that under the uncertain fat-tail prior $\mathcal{LG}(\alpha, \tilde{\xi})$ with any $\alpha > 0$, the robust strategy (28) is always a LASSO strategy. Theorem 2 provides the system of economic arguments for developing this robust strategy. Therefore, under fairly general conditions, we show that the LASSO algorithm is a Bayesian rational strategy for agents who face prior uncertainty about the fat-tail scale and optimize the max-min expected utility.

This result provides the first economic rationale for using the LASSO technique. Unlike the statistical interpretation offered by Tibshirani (1996), our economic interpretation does not invoke the heuristic MAP estimate nor require a pure and fixed Laplace prior. In our setup, agents (i.e., arbitrageurs) are *Bayesian rational* because they use the posterior mean estimate to assess all the possible states and scenarios; the LASSO strategy they choose is also *sequentially rational* because each of them applies dynamic programming to solve a two-period objective function; the LASSO strategy also qualifies as an equilibrium strategy because each agent strategically considers the best responses of other agents before choosing the strategy. Agents' prior belief is a general mixture distribution (10) that has a raw kurtosis from 3 to 6.125, depending on the mixture weight α. Our theory holds for any $\alpha \in (0, 1]$ which covers a wide range of fat-tailedness of the prior distribution.

The next proposition shows the restrictive prior assumption in the MAP-based interpretation. This does not work if the prior is not exactly Laplacian or if the prior is uncertain. We discuss later that the MAP-based trading rule not only violates Bayesian rationality but also lacks sequential rationality. As a non-equilibrium strategy, it can easily incur losses.

Proposition 2. *When $\alpha = 1$, the robust LASSO strategy (28) is observationally equivalent to a heuristic, feedback trading strategy when arbitrageurs all adopt the MAP learning rule to estimate $\tilde{v}$ under a pure and fixed Laplace prior $\mathcal{L}(0, \xi = \xi_w)$. This can be written as*

$$Z_{2,n}(y_1; \alpha = 1, K(\xi_w)) = Z_{2,n}^{\text{map}}(y_1; \alpha = 1, \xi = \xi_w) = \frac{(1 - \beta_2\lambda_2)(\widehat{v}^{\text{map}} - \lambda_1 y_1)\mathbf{1}_{|y_1| > K(\xi_w)}}{\lambda_2(N+1)}, \quad (32)$$

where the MAP estimate $\widehat{v}^{\text{map}}$ coincides with the LASSO estimate $\widehat{v}^{\text{lasso}}$ defined by Eq. (30):

$$\widehat{v}^{\text{map}}(y_1; \alpha = 1, \xi = \xi_w) = \arg\max_v \frac{f(y_1|v) f(v; \alpha = 1, \xi_w)}{f(y_1)} = \frac{\mathcal{S}(y_1; \kappa_a \sigma_u)}{\beta_1} = \widehat{v}^{\text{lasso}}(y_1; \xi_w). \quad (33)$$

When $\alpha \neq 1$, the above coincidence breaks. The MAP-based trading rule given the fixed prior $\mathcal{LG}(\alpha, \xi_w)$ differs from the robust LASSO strategy (28) given the uncertain prior $\mathcal{LG}(\alpha, \tilde{\xi})$:

$$Z_{2,n}(y_1; \alpha, K(\xi_w)) \neq Z_{2,n}^{\text{map}}(y_1; \alpha, \xi_w), \qquad \text{for any } \alpha \in (0, 1). \quad (34)$$

Proof. See Appendix A.8. □

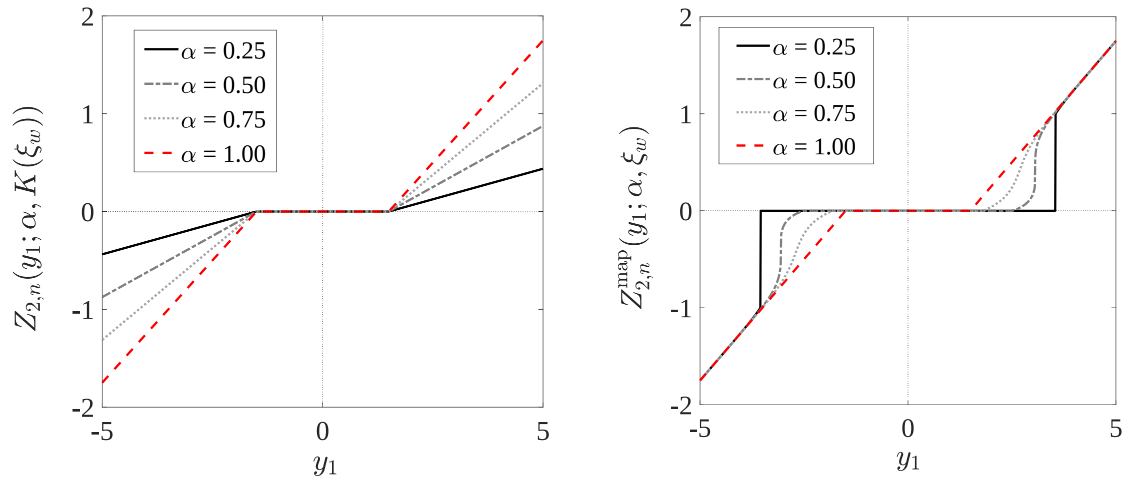

Figure 3. The robust LASSO strategy $Z_{2,n}(y_1; \alpha, K(\xi_w))$ and the MAP-based trading rule $Z_{2,n}^{\text{map}}(y_1; \alpha, \xi_w)$ for different values of α.

The MAP-based heuristic strategy is a LASSO strategy only when traders have the pure fixed Laplace prior $\mathcal{LG}(\alpha = 1, \xi)$. For any $\alpha \in (0, 1)$, we need to numerically determine the MAP strategy $Z_{2,n}^{\text{map}}$ by computing the posterior and finding its mode $\widehat{v}^{\text{map}}$. Figure 3 shows the functional profiles of the two strategies. Due to Bayesian rationality (which averages across all possibilities), the robust strategy satisfies $Z_{2,n}(y_1; \alpha, K(\xi_w)) = \alpha Z_{2,n}(y_1; \alpha = 1, K(\xi_w))$ and its threshold $K(\xi_w)$ is independent of α. In contrast, the MAP strategy lacks this scaling property: its asymptotes are independent of α and its threshold increases as α decreases.[13] Moreover, the MAP-based strategy becomes discontinuous when α is sufficiently small. It has a wider inaction region but, once triggered, tends to respond at the highest intensity. This *all-or-none* decision just relies on a heuristic threshold (following the MAP estimate) to distinguish whether the noisy input contains a Laplacian or Gaussian signal: when the input y_1 exceeds the threshold, the MAP algorithm treats y_1 as if it surely contains the Laplacian signal $\tilde{v}_L$ in Eq. (11). This kind of data classification is frequently used for machine learning tasks.[14] However, it may violate Bayesian rationality, as can be seen in our setup.

The feedback trading rule $Z_{2,n}^{\text{map}}$ does not obey the sequential rationality either. Although it implicitly assumes that no one would trade in the first (watching) period, a trader may profitably disturb the price at $t = 1$, considering that those feedback traders can misinterpret it as a true signal and overreact to her anonymous trading. In general, the MAP-based trading rule is not an equilibrium strategy.[15] It is not derived by rational agents who use backward induction to dynamically optimize their utility functions.

[13] We can show that $\lim_{y_1 \to \pm\infty} \frac{1}{y_1} Z_{2,n}^{\text{map}}(y_1; \alpha, \xi) = \frac{(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)}{\beta_1\lambda_2(N+1)}$, which is independent of α.

[14] See, for example, the *naive Bayes classifier* (Domingos and Pazzani (1997)).

[15] If there is no model risk, rational traders will settle at an equilibrium as in Theorem 1.

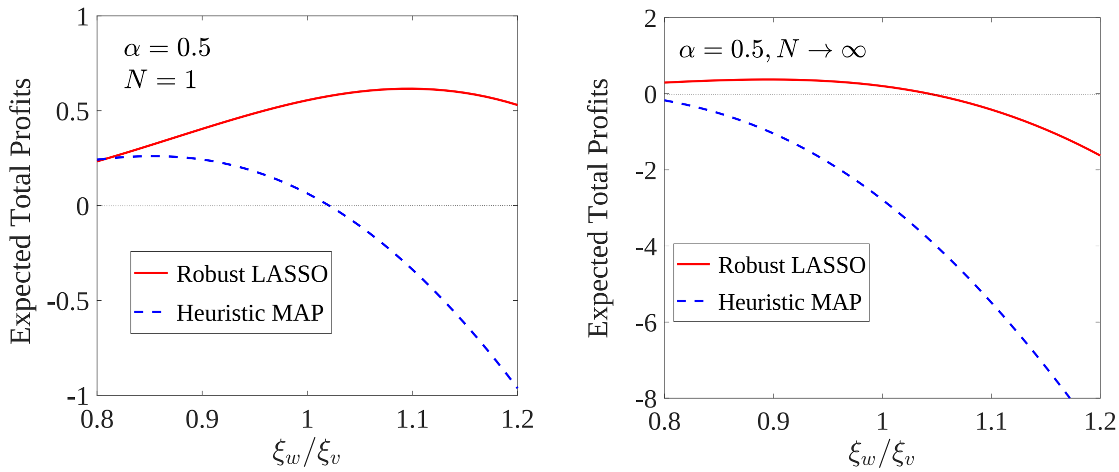

Figure 4. The expected total profits when arbitrageurs all follow the same robust LASSO strategy $Z_{2,n}(y_1; \alpha, K(\xi_w))$ versus those when they follow the MAP-based feedback trading rule $Z^{\text{map}}_{2,n}(y_1; \alpha, \xi_w)$ under the fixed prior $\mathcal{LG}(\alpha, \xi = \xi_w)$. We examine the case of $\alpha = 0.5$ and a wide range of values of ξ_w relative to the true prior value of ξ_v. The left panel reports the monopolistic result $N = 1$. The right panel is about the "competitive" limit $N \to \infty$.

For the LASSO strategy $Z_{2,n}(y_1; \alpha, K)$, its no-trade zone $[-K, K]$ does not play the role of signal classification but shrinks a range of ambiguous estimates to zero, consistent with the objective of robust optimization. It is also remarkable that all the Bayesian-rational strategies solved in Section 4 are convex for $y_1 > 0$ and concave for $y_1 > 0$. Figure 3 shows that the MAP strategy violates this rational curvature property for any $\alpha \in (0, 1)$.

As a principal, rational agents should abandon the MAP-based alternatives altogether. The MAP strategy may emerge perhaps when some traders have been educated to take "advantage" of the MAP method. It is unclear to what extend this method has been adopted by financial practitioners. The answer may well depend on their educational backgrounds. For example, a quant who was trained in the field of image analysis (e.g., Greig, Porteous, and Seheult (1989)) or speech recognition (e.g., Lim and Oppenheim (1979)) but lacks systematic training in finance or economics may have a higher tendency to embrace the MAP estimation. When a substantial fraction of quants have used it to develop their trading strategies, the market may have systemic risk and disasters like the quant meltdown in 2007 may not be uncommon. As shown in Figure 4, the expected total profits when traders all follow the Bayesian-rational LASSO strategy widely dominates those when they follow the heuristic MAP-based strategy. This dominance holds for arbitrary values of N and ξ_w. The MAP-based strategy tends to incur significant losses when traders overestimate the scale of fat tails (i.e., $\xi_w > \xi_v$). This numerical example is just suggestive. It indicates the importance of bridging the gap between neoclassical economics and machine learning technology.

5.2 Limits to arbitrage and "cartel" effect

Now we discuss the implications of the robust LASSO strategy in our model economy. The literature on limits of arbitrage (as reviewed by Gromb and Vayanos (2010)) has documented various market frictions which have a common feature as to limit arbitrageurs' *ability* to trade. Free from such frictions, our model is well suited to demonstrate a mechanism which only affects arbitrageurs' *willingness* to trade. In our setup, all the traders are risk neutral. They face no financial or trading constraints, except the price impact costs.[16] This setup highlights *model risk management* as the key friction. With uncertain fat-tail priors, this friction can lead to a robust strategy with a wide no-trade zone. There are two channels for this strategy to cause limited arbitrage. One is arbitrageurs' idleness in both the watching period ($t = 1$) and the speculation period ($t = 2$). The other channel is a subtle "cartel" effect due to arbitrageurs' conservative trading at $t = 2$.

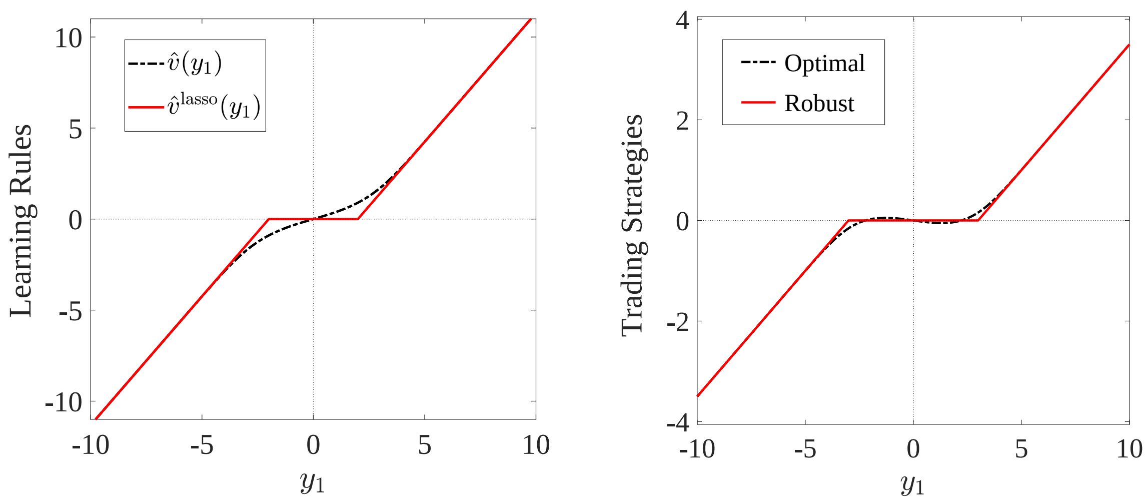

Figure 5. Left: the posterior mean estimate $\widehat{v}(y_1)$ and the LASSO estimate $\widehat{v}^{\text{lasso}}(y_1)$ under a common prior $\mathcal{LG}(\alpha, \xi_v)$. Right: the perfectly optimal strategy $Z^o_{2,n}(y_1; \alpha, \xi_v)$ and the asymptotically unbiased robust strategy $Z_{2,n}(y_1; \alpha, K(\xi_v))$.

To illustrate the first channel, we compare the unbiased robust strategy $Z_{2,n}(y_1; \alpha, K(\xi_v))$ (corresponding to the case of $\xi_w = \xi_v$) with the perfectly optimal strategy $Z^o_{2,n}(y_1; \alpha, \xi_v)$ (corresponding to the case that traders know the true prior ξ_v). By Theorem 3, the strategy $Z_{2,n}$ equivalently implements the LASSO estimate $\widehat{v}^{\text{lasso}}$,[17] whereas the optimal strategy $Z^o_{2,n}$ is directly driven by the posterior mean $\widehat{v}$. As the left panel of Figure 5 shows, the posterior mean estimate $\widehat{v}$ is a smooth nonlinear function, while the LASSO estimate $\widehat{v}^{\text{lasso}}$ is a soft-

[16] In standard Kyle-type models, the price impact costs are not at all detrimental to market efficiency.

[17] By Proposition 2, $\widehat{v}^{\text{lasso}}$ coincides with the posterior mode estimate $\widehat{v}^{\text{map}}$ only when $\alpha = 1$.

thresholding function which is zero for $y_1 \in [-\kappa_a\sigma_u, \kappa_a\sigma_u]$ and is linear outside that region. One can also see that $\widehat{v} > \widehat{v}^{\text{lasso}}$ when $y_1 > 0$ and $\widehat{v} < \widehat{v}^{\text{lasso}}$ when $y_1 < 0$. The right panel of Figure 5 compares these two strategies. Again, the optimal strategy $Z^o_{2,n}$ is a nonlinear function with asymptotic linearity, whereas the robust strategy $Z_{2,n}$ is a soft-thresholding function with an inaction zone $[-K, K]$. It can be verified that $Z_{2,n}(y_1; \alpha, K(\xi_v)) < Z^o_{2,n}(y_1; \alpha, \xi_v)$ for $|y_1| > K$. This follows from robust control which tends to produce conservative responses.

The robust LASSO strategy only reacts to large events and deliberately ignores small ones. This feature is similar to many phenomena in behavioral economics, including limited attention, status quo bias, anchoring and adjustment, among others; see Barberis and Thaler (2003) and Gabaix (2014). Despite such similarities, the LASSO strategy (28) is the rational choice by traders in our setting. They leave money on the table because the fat-tailed model risk discourages them from betting on directionally ambiguous pricing errors.

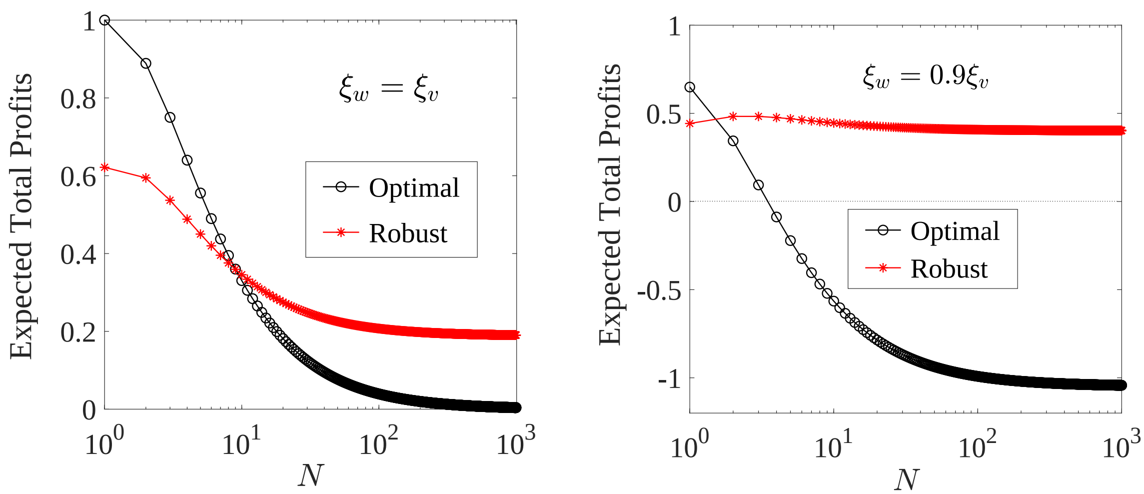

Figure 6. The expected total trading profits of all arbitrageurs when they follow the robust strategy $Z_{2,n}(y_1; \alpha, K(\xi_w))$ versus when they follow the optimal strategy $Z^o_{2,n}(y_1; \alpha, \xi_w)$.

To elucidate the second channel, we need to compare the profitability of the robust strategy $Z_{2,n}(y_1; \alpha, K(\xi_w))$ with that of the optimal strategy $Z^o_{2,n}(y_1; \alpha, \xi_w)$.[18] The comparison is on a fair ground when these strategies can converge as $|y_1| \to \infty$. Figure 6 shows the total trading profits earned by all arbitrageurs when they follow the same strategy. The left panel shows the case of $\xi_w = \xi_v$, i.e., arbitrageurs are asymptotically unbiased. With the optimal strategy, they enjoy oligopoly profits for small values of N, while their total profits decay rapidly toward zero as N increases to infinity. In contrast, arbitrageurs' total profits decay

[18]This is a benchmark strategy as if traders ignore model risk and follow the "optimal" strategy using the averaged prior ξ_w. Another benchmark is the rational-expectations strategy, $\mathrm{E}_w[Z^o_{2,n}(y_1; \alpha, \tilde{\xi})]$, which is less tractable but extremely close to $Z^o_{2,n}(y_1; \alpha, \xi_w)$. We have $\mathrm{E}_w[Z^o_{2,n}(y_1; \alpha, \tilde{\xi})] \to Z^o_{2,n}(y_1; \alpha, \xi_w)$ as $|y_1| \to \infty$.

slowly when they follow the robust LASSO strategy. For $N \leq 10$, this strategy allows them to capture the majority of what they would earn using the optimal strategy. For $N > 10$, their robust trading profits even surpass the optimal trading profits. In the limit $N \to \infty$, their total profit converges to a level at about 20% of the maximal monopoly profit (normalized to one). This result may be surprising given that the difference between these two strategies vanishes everywhere: $\lim_{N\to\infty} |Z_{2,n}(y_1; \alpha, K(\xi_w)) - Z^o_{2,n}(y_1; \alpha, \xi_w)| = 0$.

Figure 6 also compares the total trading profits when $\xi_w = 0.9\xi_v$. With the robust strategy, arbitrageurs earn significant profits which are almost flat as N increases. In contrast, the optimal strategy results in more and more losses when N increases from 4 to infinity. Its performance is much more sensitive to the competition. From extensive numerical experiments, we find that the robust strategy can often outperform the optimal strategy when they are both biased (i.e., $\xi_w \neq \xi_v$). Thus, the subjective optimal strategy seems unattractive in the presence of estimation bias and trading competition.

Theorem 4. *In the symmetric equilibrium of Theorem 1, the expectation of arbitrageurs' aggregate profit when they all follow the subjective optimal strategy $Z^o_{2,n}(y_1; \alpha, \xi_w)$ is*

$$\mathrm{E}\left[\sum_{n=1}^{N}(\tilde{v} - \tilde{p}_2) Z^o_{2,n}\right] = \frac{N(1-\beta_2\lambda_2)^2}{(N+1)\lambda_2}\mathrm{E}\left[\widehat{\theta}(\tilde{y}_1; \alpha, \xi_v)\widehat{\theta}(\tilde{y}_1; \alpha, \xi_w) - \frac{N}{N+1}\widehat{\theta}(\tilde{y}_1; \alpha, \xi_w)^2\right], \quad (35)$$

where $\widehat{\theta}(y_1; \alpha, \xi_w) := \mathrm{E}[\tilde{v} - \tilde{p}_1 | y_1; \alpha, \xi_w] = \widehat{v}(y_1; \alpha, \xi_w) - \lambda_1 y_1$ and $\widehat{v}(y_1)$ is given by Eq. (20). When $\xi_w = \xi_v$ and $N = 1$, we obtain the maximal monopoly profit, $\frac{(1-\beta_2\lambda_2)^2}{4\lambda_2}\mathrm{E}[\widehat{\theta}(\tilde{y}_1; \alpha, \xi_w)^2]$. When $\xi_w = \xi_v$ and $N \to \infty$, arbitrageurs will compete away their aggregate trading profit.

In the symmetric equilibrium of Theorem 2, the expectation of arbitrageurs' aggregate trading profit when they all follow the robust strategy $Z_{2,n}(y_1; \alpha, K(\xi_w))$ is given by

$$\mathrm{E}\left[\sum_{n=1}^{N}(\tilde{v} - \tilde{p}_2) Z_{2,n}\right] = \frac{N\alpha^2(1-\beta_2\lambda_2)^2}{(N+1)\lambda_2}\mathrm{E}\left[(\widehat{v}(\tilde{y}_1; \xi_v) - \widehat{v}^{\text{lasso}}(\tilde{y}_1; \xi_w))\widehat{\theta}^{\text{lasso}} + \frac{(\widehat{\theta}^{\text{lasso}})^2}{N+1}\right]. \quad (36)$$

When $0 < \xi_w \leq \xi_v$, the expected aggregate profit is always positive and has a positive limit:

$$\lim_{N\to\infty} \mathrm{E}\left[\sum_{n=1}^{N}(\tilde{v} - \tilde{p}_2) Z_{2,n}\right] = \frac{\alpha^2(1-\beta_2\lambda_2)^2}{\lambda_2}\mathrm{E}[(\widehat{v}(\tilde{y}_1; \xi_v) - \widehat{v}^{\text{lasso}}(\tilde{y}_1; \xi_w))\widehat{\theta}^{\text{lasso}}(\tilde{y}_1; \xi_w)] > 0, \quad (37)$$

because $\widehat{v}(\tilde{y}_1; \xi_v) - \widehat{v}^{\text{lasso}}(\tilde{y}_1; \xi_w)$ and $\widehat{\theta}^{\text{lasso}}(\tilde{y}_1; \xi_w)$ bear the same signs when $|\tilde{y}_1| > K(\xi_w)$.

Proof. See Appendix A.9. Note that we have used $\widehat{v}(\tilde{y}_1; \xi_v)$ to stand for $\widehat{v}(\tilde{y}_1; \alpha = 1, \xi_v)$. □

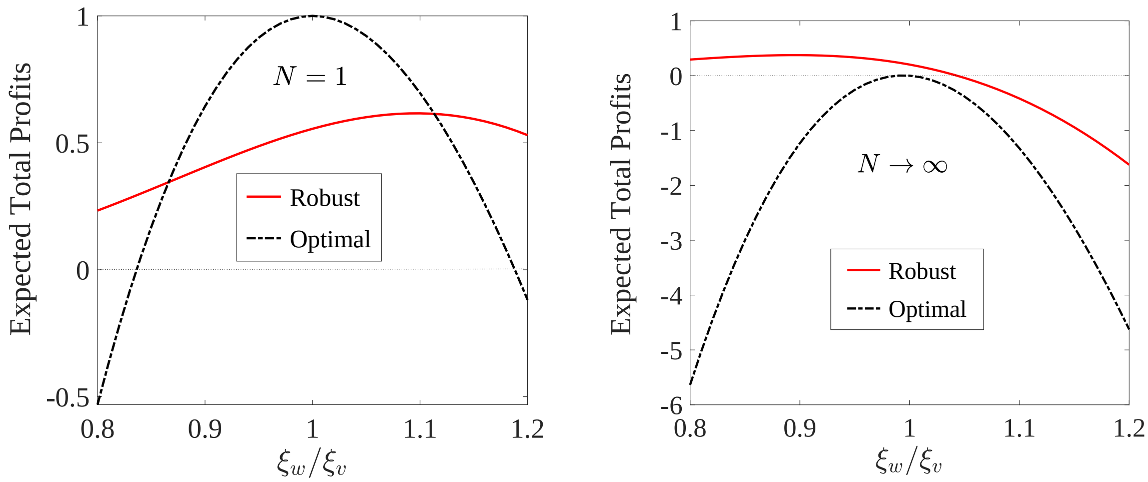

Figure 7. The profitability of the robust strategy $Z_{2,n}(\tilde{y}_1; \alpha, K(\xi_w))$ versus that of the optimal strategy $Z^o_{2,n}(\tilde{y}_1; \alpha, \xi_w)$ for a range of values of ξ_w relative to the true prior ξ_v.

Figure 7 compares the profitability of these two strategies for different values of ξ_w in two extreme cases. In the monopoly case ($N = 1$), a single arbitrageur earns the maximal profit if she use the optimal strategy when $\xi_w = \xi_v$. Her expected profit is, however, sensitive to the bias and becomes negative when $|\xi_w - \xi_v|/\xi_v \gtrsim 20\%$. In contrast, the performance of the robust strategy is less sensitive to the bias and remains positive for a much wider range of ξ_w. By following the price trends the robust LASSO strategy is more likely to trade on the right side, whereas the optimal strategy can bet on the wrong side much more frequently. In the competitive case ($N \to \infty$), the total arbitrage profit under the optimal strategy is almost always negative unless $\xi_w \approx \xi_v$. This contrasts with the robust strategy which maintains positive profitability for $\xi_w \leq \xi_v$. While both strategies may lose money for $\xi_w > \xi_v$, the robust strategy lose much less. Our results share some similarities with the findings of Zhu and Zhou (2009) who report that the technical trading rules can be robust to model specification and tend to substantially outperform the seemingly optimal trading strategies under model uncertainty.

How can an infinite number of traders make significant profits when they follow the same strategy? Figure 8 (left) plots $[\widehat{v}(y_1; \xi_v) - \widehat{v}^{\text{lasso}}(y_1; \xi_w)] \cdot \widehat{\theta}^{\text{lasso}}(y_1; \xi_w)$ as a function of the input y_1 when $\xi_w = \xi_v$. This product term drives the positive limit of Eq. (37). It shows two sharp peaks in the outskirts of no-trade zone. Recall that the LASSO estimate shrinks the mean prediction (Figure 5): $|\widehat{v}^{\text{lasso}}(y_1)| < |\widehat{v}(y_1)|$. As this shrinkage is independent of N, it can actually benefit an arbitrary number of traders such that the entire group of them can buy (or sell) the asset on average below (or above) the fair price; see the right panel of Figure 8. Consequently, this *statistical arbitrage* remains profitable even when $N \to \infty$.

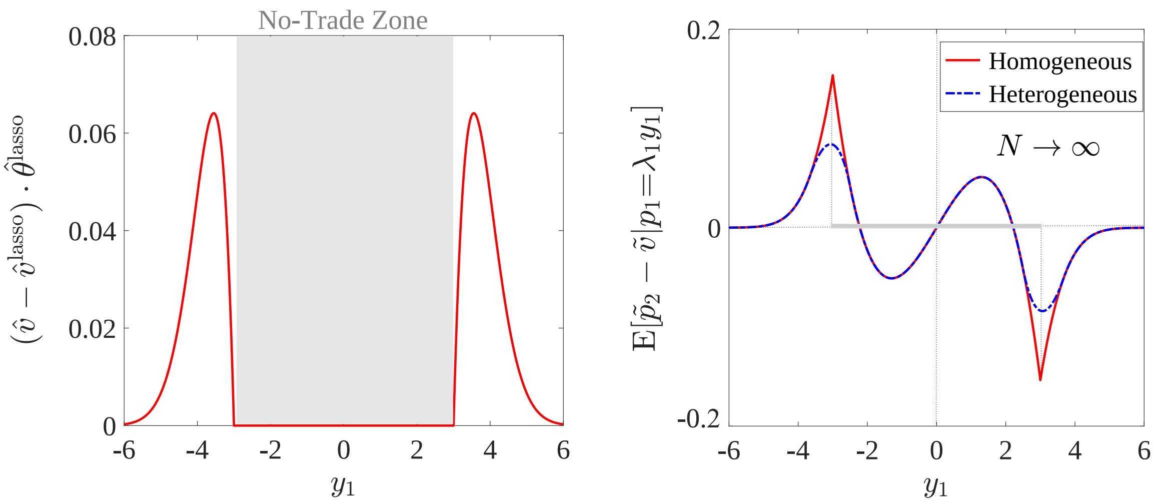

Figure 8. Left: $[\widehat{v}(y_1; \xi_v) - \widehat{v}^{\text{lasso}}(y_1; \xi_w)] \cdot \widehat{\theta}^{\text{lasso}}(y_1; \xi_w)$ as a function of y_1 when $\xi_w = \xi_v$. Right: $\mathrm{E}[\tilde{p}_2 - \tilde{v} | p_1 = \lambda_1 y_1]$ versus y_1, where the shaded segment indicates the no-trade zone. The red solid line is when arbitrageurs follow the robust strategy with the same threshold $K(\xi_v)$. The blue dashed line is when they use heterogeneous thresholds, denoted $K(\xi_{w,n})$. We impose $\frac{1}{N}\sum_{n=1}^{N} \xi_{w,n}^{-1} = \xi_v^{-1}$ to facilitate a fair comparison.

The non-vanishing profit earned by arbitrageurs can be attributed to their conservative trading outside the no-trade zone. When $\xi_w \le \xi_v$, the robust LASSO strategy always trades less than the unbiased optimal strategy: $|Z_{2,n}(y_1; \alpha, K(\xi_w))| < |Z_{2,n}^{o}(y_1; \alpha, \xi_v)|$. This under-trading mitigates their competition, allowing traders to accumulate extra market power that facilitates a "cartel" to protect their profits. Eq. (29) allows us to rewrite Eq. (37) as

$$\frac{\alpha^2(1-\beta_2\lambda_2)^2}{\lambda_2}\mathrm{E}[(\widehat{v} - \widehat{v}^{\text{lasso}})\widehat{\theta}^{\text{lasso}}] = 2\alpha(1-\beta_2\lambda_2)\mathrm{E}[(\tilde{v} - \widehat{v}^{\text{lasso}}) \cdot Z_{2,n}(\tilde{y}_1; \alpha, \xi_w, N=1)], \quad (38)$$

equivalent to the monopoly case that an arbitrageur pays the cheaper price $\widehat{v}^{\text{lasso}}$ to receive $\tilde{v}$. This seemingly collusive outcome is not due to any trading frictions or financial constraints. Like tacit collusion, it requires no communication device or explicit agreements. The "cartel" is facilitated by traders' strategic exercise of robust optimization. By rewarding arbitrageurs, this novel effect serves as another channel that inevitably impedes market efficiency.

Price efficiency can be fully restored at $t = 2$ if the economy hosts an infinite number of arbitrageurs and all of them follow the unbiased optimal strategy. Nonetheless, if arbitrageurs are constrained by model risks and all adopt the robust LASSO strategy, then there will be persistent pricing errors in the neighborhoods of $p_1 = \pm\lambda_1 K$, as shown in the right panel of Figure 8. In general, whenever a mass of them are constrained by model risks, they will trade conservatively, amass extra market power, and sustain inefficient prices.

Proposition 3. *Suppose the economy hosts an infinite number of risk-neutral arbitrageurs ($N \to \infty$). If there is ever a finite measure $\phi \in (0, 1]$ of them constrained by model risk as in Theorem 2, then the asset price $\tilde{p}_2$ is inefficient for almost all realizations of $\tilde{p}_1 = p_0 + \lambda_1 \tilde{y}_1$:*

$$\mathrm{E}[\tilde{p}_2 - \tilde{v}|\tilde{p}_1] \to \alpha(1 - \beta_2\lambda_2) \left\{ [\widehat{v}(\tilde{y}_1; \xi_w) - \widehat{v}(\tilde{y}_1; \xi_v)] + \phi \left[\widehat{\theta}^{\text{lasso}}(\tilde{y}_1; \xi_w) - \widehat{\theta}(\tilde{y}_1; \xi_w) \right] \right\} \neq 0. \quad (39)$$

Proof. See Appendix A.10. Note that we have used $\widehat{\theta}(\tilde{y}_1; \xi_w)$ to stand for $\widehat{\theta}(\tilde{y}_1; \alpha = 1, \xi_w)$. □

Eq. (39) shows two sources for inefficient prices at $t = 2$. One is the estimation bias (i.e., $\xi_w \neq \xi_v$) which is applicable to all traders. The other is the under-trading by a fraction of traders who exercise robust control. The price is inefficient almost everywhere, unless all the arbitrageurs know the true prior (i.e., $\xi_w = \xi_v$ and $\phi = 0$). Bossaerts et al. (2010) and Ahn et al. (2014) document considerable heterogeneity in risk and ambiguity aversion and find that a fraction of individuals' behavior is consistent with the standard expected utility. Proposition 3 shows that our result is robust to investors' heterogeneous preferences.

This implication also holds when arbitrageurs use the LASSO strategy (28) with heterogeneous thresholds, denoted $K(\xi_{w,n})$ for $n = 1, ..., N$. The aggregate robust trading with heterogeneous thresholds becomes a smoother function of y_1 (Figure 16 in Appendix A.10). The right panel of Figure 8 shows that heterogeneous thresholds can partially smooth out the "cartel" effect but cannot restore price efficiency.

6 Extensions and Applications

6.1 Uncertainty about the frequency of fat-tail shocks

Our setup can be easily extended to the case that traders are uncertain about both prior parameters, α and ξ_v. In other words, they face model risk about the frequency and magnitude of mispricings simultaneously. Under the Gaussian-Laplacian mixture distribution (10), the *fat-tailedness* (as measured by the raw kurtosis) of $\tilde{v}$ is given by $3 + 3\frac{(4+\alpha)(1-\alpha)}{(2-\alpha)^2} \in (3, \frac{49}{8}]$, which is a simple function of α. So we are considering a general situation where traders face uncertainty about not only the stochastic volatility but also the fat-tailedness of stock value.

Proposition 4. *Suppose arbitrageurs are uncertain about both α and ξ_v, with common priors denoted as $\mathcal{LG}(\tilde{\alpha}, \tilde{\xi})$ where $\tilde{\alpha} \in [\alpha_L, \alpha_H]$ and as before $\tilde{\xi} \in [\xi_L, \xi_H]$. If they are asymptotically ambiguity-neutral as (C1) in Theorem 2 and if the other two conditions (C2) and (C3) also hold, then there exists a symmetric debiased equilibrium where arbitrageurs watch the market* *without trading at $t = 1$ and they follow a robust LASSO strategy at $t = 2$:*

$$Z_{2,n}(y_1; \overline{\alpha}, K(\xi_w)) = \frac{\overline{\alpha}(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)}{\beta_1\lambda_2(N+1)}\mathcal{S}(y_1; K) = \frac{\overline{\alpha}(1-\beta_2\lambda_2)}{\lambda_2(N+1)} \cdot \widehat{\theta}^{\text{lasso}}(y_1; K), \tag{40}$$

where $\overline{\alpha} := \mathrm{E}[\tilde{\alpha}]$ is the prior mean of $\tilde{\alpha}$ and $\widehat{\theta}^{\text{lasso}}(y_1; K)$ is still defined by Eq. (31).

Proof. Eq. (40) follows from Theorem 2 and the scaling property that α can be factored out from the fixed-prior solution (28). The more rigorous proof resembles Appendix A.6. □

Recall from Figure 3 that the robust LASSO strategy has the same threshold $K(\xi_w)$ for any positive values of α. With Bayesian learning, traders average across all possible priors. They respond with $\overline{\alpha} = \mathrm{E}[\tilde{\alpha}]$ in Eq. (40), since $\mathrm{E}[Z_{2,n}(y_1; \tilde{\alpha}, K)] = \mathrm{E}[\tilde{\alpha}]Z_{2,n}(y_1; \alpha = 1, K)$. This also explains why our original setup only focuses on the prior uncertainty about ξ_v.

Thus, our derivation of the LASSO strategy holds in the general situation where agents have formed a mixture prior on the estimated parameter but know little about the frequency or the scale of its fat-tail component. This general applicability is in sharp contrast with the MAP-based interpretation which requires a pure and fixed Laplace prior.

6.2 LASSO for a long-short portfolio of mispriced stocks

The result below rationalizes the application of a vector form of the LASSO algorithm. It is a technical strategy that forecasts and exploits multiple misvalued stocks simultaneously.

Proposition 5. *Suppose arbitrageurs anticipate multiple independent assets to be mispriced. Each asset, indexed by $j = 1, ..., J$, is traded in a two-period environment with linear price movements as in (12) and aggregate order flows as in (16). Arbitrageurs have uncertain priors on each asset, denoted $\tilde{v}_j \sim \mathcal{LG}(\tilde{\alpha}_j, \tilde{\xi}_j)$. If (C1)-(C3) in Theorem 2 hold, there is an equilibrium where arbitrageurs choose to idle at $t = 1$ and follow a LASSO strategy at $t = 2$:*

$$\mathbf{Z}_{2,n}(\mathbf{p}_1) := \left\{ \frac{\overline{\alpha}_j(1-\beta_{2,j}\lambda_{2,j})}{\lambda_{2,j}(N+1)} \cdot \widehat{\theta}_j^{\text{lasso}}(p_{1,j}; \xi_{w,j}) : j = 1, ..., J \right\}, \tag{41}$$

where $\overline{\alpha} = \mathrm{E}^j[\tilde{\alpha}_j]$ is the prior mean frequency and $\xi_{w,j}$ is the weighted harmonic mean of $\tilde{\xi}_j$. The vector of LASSO estimates, $\widehat{\mathbf{\Theta}}^{\text{lasso}} = \{\widehat{\theta}_j^{\text{lasso}}(p_{1,j}; \xi_{w,j}) : j = 1, ..., J\}$, is defined by

$$\widehat{\mathbf{\Theta}}^{\text{lasso}} := \underset{\{\theta_1,\dots,\theta_J\}}{\arg\min} \sum_{j=1}^{J} \left\{ \frac{1}{2} \left| p_{1,j} - \frac{\beta_{1,j}\lambda_{1,j}}{1-\beta_{1,j}\lambda_{1,j}}\theta_j \right|^2 + \left(\frac{\lambda_{1,j}\sigma_{u,j}}{1-\beta_{1,j}\lambda_{1,j}} \right)^2 \frac{|\theta_j|}{\xi_{w,j}} \right\}. \tag{42}$$

Proof. See Appendix A.11. This is based on Theorem 2, Theorem 3, and Proposition 4. □

Proposition 5 can give rational interpretations of certain technical or algorithmic trading rules. These are triggered to trade whenever stock prices hit across predefined price levels (Lo, Mamaysky, and Wang (2000); Zhu and Zhou (2009); Han, Liu, Zhou, and Zhu (2021)). At first glance, such mechanical plans are at odds with Bayesian rationality. Proposition 5 suggests that simple trading rules for constructing a long-short portfolio may well be the solution of sophisticated risk management. This argument agrees with Zhu and Zhou (2009) who show that the widely used moving average trading rule can add value to asset allocation under uncertainty about predictability or about the true model governing the stock price.

Proposition 5 may also rationalize positive feedback traders who extrapolate and chase price trends, as discussed in the behavioral literature (DeLong et al. (1990), Barberis et al. (2015, 2018)). The multi-asset LASSO strategy (41) shows a similar extrapolative feature. Its two momentum "arms" are ready to long and short equities as their prices move beyond endogenous no-trade zones. This strategy only respond to most recent winners and losers.

6.3 Sparse predictors in the cross-section of stock returns

When a market lacks weak-form efficiency, we may predict stock returns based on historical stock prices and trading volumes. This is not only about the prediction of a stock by using its own trading data (as in Theorem 2 or Proposition 5) but also applicable to cross-sectional return predictions (as in Chinco et al. (2019)). Here, we briefly extend our theory to the empirical quest of Chinco et al. (2019). Our discussion may provide some intuition about the return predictability and the usefulness of LASSO in their paper. We also propose that the LASSO may be used to filter predictive stock returns for robustness.

Consider a two-period market where a large number of stocks are traded. Some stocks have fat tails which may be correlated with the fat tails of other stocks. Like Chinco et al. (2019), we consider the task to predict the next-minute stock returns using the entire cross-section return data over a short-term window, denoted as $\{r_{1,j} : j = 1, ..., J\}$. To be specific, we attempt to forecast stock k's next-minute return, $\tilde{r}_{2,k}$. There are perhaps a small number of stocks that have predictive power for $\tilde{r}_{2,k}$. Let $\mathbf{S}_k \coloneqq \{j : \tilde{r}_{1,j} \text{ is informative about } \tilde{r}_{2,k}\}$ be the subset of such stocks (predictors). For simplicity, suppose we have learned about the subset $\mathbf{S}_k$. This is actually achieved in Chinco et al. (2019) by applying the LASSO to select the subset of predictors. Suppose the economy operates in a way that the mispricing of stock k is approximately a linear combination of the mispricings of stocks in the subset $\mathbf{S}_k$:

$$\tilde{\theta}_k \approx \sum_{j\in\mathbf{S}_k} \Omega_{k,j}\tilde{\theta}_j = \sum_{j\in\mathbf{S}_k} \Omega_{k,j}(\tilde{v}_j - p_{1,j}), \tag{43}$$

where $\Omega_{k,j}$ reflects the degree of predictive power of stock j on stock k. Eq. (43) serves as an assumption. The microfoundation is beyond the scope of this paper. Correlated fat-tail mispricings might arise from correlated news shocks or from asynchronous trading on dispersed private signals drawn from a multivariate fat-tail distribution.

As before, our prior belief for each stock $j \in \mathbf{S}_k$ is $\tilde{v}_j \sim \mathcal{LG}(\alpha_j, \tilde{\xi}_j)$ where $\tilde{\xi}_j$ represents our prior uncertainty about the scale of fat-tail shocks. If the conditions for Theorem 2 hold for each stock, then we can apply the LASSO estimate for each $\tilde{\theta}_j$ (by Eq. (31)) and add them together to form a robust estimate of the pricing error $\tilde{\theta}_k$:

$$\widehat{\theta}_k^{\text{rob}} := \sum_{j\in\mathbf{S}_k} \Omega_{k,j}\widehat{\theta}_j^{\text{lasso}} = \sum_{j\in\mathbf{S}_k} \Omega_{k,j}\alpha_j \frac{1-\beta_{1,j}\lambda_{1,j}}{\beta_{1,j}} \left[y_{1,j} - \text{sign}(y_{1,j})K_j(\xi_{w,j})\right] \mathbf{1}_{|y_{1,j}|>K_j(\xi_{w,j})}. \quad (44)$$

In general, $\widehat{\theta}_k^{\text{rob}}$ is not exactly the LASSO estimate of $\tilde{\theta}_k$, because the threshold K_j can be different across predictors, preventing $\widehat{\theta}_k^{\text{rob}}$ from being a simple soft-thresholding function.

If we normalize the initial stock price to be one (i.e., $p_{0,j} = 1$) for each stock $j \in \mathbf{S}_k$, then the lagged return for stock j can be written as $r_{1,j} = (p_{1,j} - p_{0,j})/p_{0,j} = p_{1,j} - 1 = \lambda_{1,j}y_{1,j}$. Suppose the short-term cross-sectional return predictability has not been exploited by any traders (e.g., prior to the publication of Chinco et al. (2019)). Then, the order flows for each stock can be described by Eq. (13). For stock k, the next-minute return is

$$\tilde{r}_{2,k} := \frac{\tilde{p}_{2,k} - p_{1,k}}{p_{1,k}} = \frac{\lambda_{2,k}\tilde{y}_{2,k}}{p_{1,k}} = \frac{\lambda_{2,k}(\beta_{2,k}\tilde{\theta}_k + \tilde{u}_k)}{1 + r_{1,k}} \approx \frac{\lambda_{2,k}\beta_{2,k}\sum_{j\in\mathbf{S}_k}\Omega_{k,j}\tilde{\theta}_j + \lambda_{2,k}\tilde{u}_k}{1 + r_{1,k}}. \quad (45)$$

The posterior mean estimate of $\tilde{r}_{2,k}$ is

$$\widehat{r}_{2,k} := \mathrm{E}\left[\tilde{r}_{2,k}|\{r_{1,j} : j \in \mathbf{S}_k\}\right] = \frac{\lambda_{2,k}\beta_{2,k}}{1 + r_{1,k}} \sum_{j\in\mathbf{S}_k} \Omega_{k,j} \int w_j(\xi)\mathrm{E}\left[\tilde{\theta}_j \middle| r_{1,j}; \alpha_j, \tilde{\xi}_j = \xi\right] d\xi, \quad (46)$$

which is a complicated nonlinear function of the vector $\{r_{1,j} : j \in \mathbf{S}_k\}$. To make a robust estimate of $\tilde{r}_{2,k}$, we can replace the above posterior means with the LASSO estimates:

$$\widehat{r}_{2,k}^{\text{rob}} := \frac{\lambda_{2,k}\beta_{2,k}}{1 + r_{1,k}} \sum_{j\in\mathbf{S}_k} \Omega_{k,j}\widehat{\theta}_j^{\text{lasso}}(r_j; \alpha_j, K_j(\xi_{w,j})) = a_k + \sum_{j\in\mathbf{S}_k, |r_{1,j}|>\lambda_{1,j}K_j} b_{k,j}r_{1,j}. \quad (47)$$

This is a simple linear function of the lagged stock returns (predictors), where

$$a_k := -\sum_{j\in\mathbf{S}_k, |r_{1,j}|>\lambda_{1,j}K_j} b_{k,j}\lambda_{1,j}K_j\text{sign}(r_{1,j}) \quad \text{and} \quad b_{k,j} := \frac{\lambda_{2,k}\beta_{2,k}}{1 + r_{1,k}}\Omega_{k,j}\alpha_j\frac{1-\beta_{1,j}\lambda_{1,j}}{\lambda_{1,j}\beta_{1,j}}. \quad (48)$$

Eq. (47) expresses both sparsity and linearity of predictive signals in the cross-section of stock returns. The sparsity comes from two selections, $j \in \mathbf{S}_k$ and $|r_{1,j}| > \lambda_{1,j} K_j$. This may help us understand the empirical results of Chinco et al. (2019) who focus on the first selection $\mathbf{S}_k$ by using the LASSO to pick up the nontrivial coefficients $b_{k,j}$ of all candidate predictors $\{r_{1,j} : j = 1, ..., J\}$ where J is a large number. Their variable selection ($j \in \mathbf{S}_k$) is different from our filtering ($|r_{1,j}| > \lambda_{1,j} K_j$) on predictors, although both have involved the LASSO estimates. Our argument of robust optimization will effectively impose the filtering threshold $\lambda_{1,j} K_j$ on the observed predictive stock returns. Here, $\lambda_{1,j}$ reflects the transaction cost per unit of order flow, while K_j is the endogenous threshold that appears in the LASSO estimation of the residual signal $\tilde{\theta}_j$.[19] The filtering effect in our theory can also "sparsify" the selection of return predictors. Therefore, we propose an application of LASSO to directly filter the cross-section of stock returns. Such a procedure may improve the robustness and even the performance in the prediction task of Chinco et al. (2019).

6.4 Ridge regression from a Gaussian mixture prior

In the statistical interpretation of LASSO, the l_1 penalty term is inserted into the objective function by the MAP estimation with a pure Laplace prior. In our interpretation, the effect of l_1 penalty is imposed by the max-min decision criteria and the curvature condition (C2) in Theorem 2. (C2) helps regularize the robust optimization problem by requiring admissible strategies to preserve the Bayesian-rational curvature property of Corollary 4.1(b). In fact, the key assumption in our theory is the fat-tailed prior distribution. If we replace this assumption with a Gaussian mixture prior and keep everything else equal, then the robust strategy under the max-min criteria is not the LASSO but the *ridge regression* which contains an l_2 penalty in its objective function. As a result, it uniformly shrinks all coefficients without sending any of them to zero.

Proposition 6. *If, instead, arbitrageurs have a mixture Gaussian prior about the asset liquidation value, then their equilibrium strategy, either optimal or robust, is equivalent to the ridge regression which is a linear function of the input y_1 without any finite inaction region.*

Proof. See Appendix A.12. □

The conditions (C1)-(C3) in Theorem 2 are not needed in Proposition 6 any more. These conditions are inferred from the existence of a *debiased* equilibrium when traders have uncertain fat-tail priors. If we remove the assumption of fat tails, we are back to the Gaussian

[19] Chinco et al. (2019) have taken into account the bid-ask spread in their LASSO-implied strategy. This spread cost alone may not fully capture the filtering effect of $\lambda_{1,j} K_j$ derived from our robust optimization.

world where learning problems are usually linear. Thus, Proposition 6 serves as a control to demonstrate that the fat-tail distribution is the key assumption in this paper.

Ridge regression is another basic and popular machine learning method. Its l_2 penalty has been integrated in other techniques. For example, the *elastic net* is a regularized regression that linearly combines the l_1 and l_2 penalties of the LASSO and ridge methods (Hastie et al. (2009)). While Proposition 6 seems to achieve another subject in rationalizing the ridge regression, we want to focus on the LASSO in this paper. It is left for future work to understand the rationality (if any) underneath many other machine learning techniques.

7 Conclusion

Machine learning seems an inevitable trend in the era of big data. Are machine learning methods heuristic approximations or rational choices for economic agents? This paper rationalizes one of the most widely used machine learning method, the LASSO algorithm. Unlike the interpretation proposed by Tibshirani (1996), our rationalization of LASSO does not invoke the heuristic MAP (i.e., the posterior mode) estimation or hinge on the restrictive assumption of a pure fixed Laplace prior. In our setup, agents (i.e., arbitrageurs) consistently use the Bayesian-rational learning (i.e., the posterior mean estimate) to evaluate all possible states. They have sequential rationality by using dynamic programming to solve their well-defined max-min expected utility function. Under general fat-tailed model risks, their robust strategy is a LASSO strategy. We also show that this robust LASSO strategy can reduce traders' competition even when the number of them becomes infinite. This induces a seemingly collusive outcome as traders' aggregate profit does not vanish even in the "competitive" limit. This is a novel mechanism for limited arbitrage.

This paper provides a theoretical demonstration that brings a popular machine learning method within the framework of neoclassic theory of financial economics. This may not be a unique example, considering that there are a set of variants developed from or related to the original LASSO. It calls for more interdisciplinary studies like this paper to develop a better understanding of the economic rationales and implications of other widely used techniques.

A Appendix

A.1 Example of Microfoundation

Here, we discuss one microfoundation for the trading environment which can serve as the background of our model setup. Consider a two-period model of Kyle (1985) with multiple ($M \geq 1$) informed traders as extended by Holden and Subrahmanyam (1992). Suppose all market participants (informed traders and market makers) hold the common knowledge that the stock liquidation value is normally distributed. Then there exists a unique subgame perfect linear equilibrium, based on the general procedures of Proposition 1 in Holden and Subrahmanyam (1992). We use the same notation $\Sigma_t := \text{Var}(\tilde{v}_0|\tilde{p}_t, \tilde{p}_{t-1}, ...\tilde{p}_0)$ for $t \in \{0, 1, 2\}$, which is the posterior variance of $\tilde{v}_0$ conditional on the price history up to time t. There are competitive market makers who will accommodate the following aggregate order flows:

$$\tilde{y}_1 = \sum_{m=1}^{M} \tilde{x}_{m,1} + \tilde{u}_1 = \beta_1(\tilde{v} - p_0) + \tilde{u}_1, \tag{A1}$$

$$\tilde{y}_2 = \sum_{m=1}^{M} \tilde{x}_{m,2} + \tilde{u}_2 = \beta_2(\tilde{v} - \tilde{p}_1) + \tilde{u}_2, \tag{A2}$$

where $\tilde{x}_{m,t}$ denotes the order placed by the m-th informed trader at time t and β_t represents the aggregate informed trading intensity at time t. In the conjectured linear equilibrium, market makers follow the linear pricing strategies below:

$$\tilde{p}_1 = p_0 + \lambda_1 \left(\sum_{m=1}^{M} \tilde{x}_{m,1} + \tilde{u}_1\right) = p_0 + \lambda_1 \tilde{y}_1, \tag{A3}$$

$$\tilde{p}_2 = \tilde{p}_1 + \lambda_2 \left(\sum_{m=1}^{M} \tilde{x}_{m,2} + \tilde{u}_2\right) = \tilde{p}_1 + \lambda_2 \tilde{y}_2, \tag{A4}$$

where the pricing coefficients (Kyle lambdas) are given by

$$\lambda_1 = \frac{\beta_1 \Sigma_1}{\sigma_u^2}, \qquad \lambda_2 = \frac{\beta_2 \Sigma_2}{\gamma \sigma_u^2}, \tag{A5}$$

As a boundary condition, it is easy to derive the total intensity of informed trading at $t = 2$:

$$\beta_2 = \frac{M}{\lambda_2 (M+1)}. \tag{A6}$$

By backward induction, we can further derive the total intensity of informed trading at $t = 1$,

$$\beta_1 = \frac{\delta M (M+1)^2 - 2M}{\lambda_1 [\delta (M+1)^3 - 2M]}, \tag{A7}$$

where $\delta := \lambda_2/\lambda_1$ is the ratio of Kyle lambdas. These results imply that

$$1 - \beta_1\lambda_1 = \frac{\delta(M+1)^2}{\delta(M+1)^3 - 2M}, \qquad 1 - \beta_2\lambda_2 = \frac{1}{M+1}.$$

The optimal trading strategy for each informed trader is

$$\tilde{x}_{m,1} = \frac{\beta_1}{M}(\tilde{v} - p_0) = \frac{\delta(M+1)^2 - 2}{\delta(M+1)^3 - 2M} \cdot \frac{\tilde{v} - p_0}{\lambda_1}, \tag{A8}$$

$$\tilde{x}_{m,2} = \frac{\beta_2}{M}(\tilde{v} - p_1) = \frac{\tilde{v} - \tilde{p}_1}{\lambda_2(M+1)}. \tag{A9}$$

The Bayesian update of the posterior variance about $\tilde{v}$ follows

$$\Sigma_2 = (1 - \beta_2\lambda_2)\Sigma_1 = \frac{\Sigma_1}{M+1}, \qquad \Sigma_1 = (1 - \beta_1\lambda_1)\Sigma_0 = \frac{\delta(M+1)^2\sigma_v^2}{\delta(M+1)^3 - 2M}. \tag{A10}$$

Combining (A5), (A6), (A7), (A8), (A9), and (A10), we find the expressions of λ_1 and λ_2:

$$\lambda_1 = \frac{\sqrt{\delta M(M+1)^2(\delta(M+1)^2 - 2)}}{\delta(M+1)^3 - 2M} \cdot \frac{\sigma_v}{\sigma_u}, \tag{A11}$$

$$\lambda_2 = \delta\lambda_1 = \sqrt{\frac{\delta M/\gamma}{\delta(M+1)^3 - 2M}} \cdot \frac{\sigma_v}{\sigma_u}. \tag{A12}$$

In order to have $\lambda_1 > 0$ and $\lambda_2 > 0$, it is equivalent to impose $\delta(M+1)^2 > 2$. This will guarantee that the denominators in (A11) and (A12) are strictly positive, $\delta(M+1)^3 - 2M > 0$. These allow us to rewrite the ratio of Kyle lambdas which is equal to δ by definition:

$$\frac{\lambda_2^2}{\lambda_1^2} = \frac{\delta(M+1)^3 - 2M}{\delta\gamma(M+1)^4 - 2\gamma(M+1)^2} = \delta^2, \tag{A13}$$

By Eq. (A13), the equilibrium ratio $\delta = \delta(M, \gamma)$ must satisfy the cubic equation:

$$(M+1)^4\gamma\delta^3 - 2(M+1)^2\gamma\delta^2 - (M+1)^3\delta + 2M = 0, \quad \text{s.t.} \quad \delta(M+1)^2 > 2. \tag{A14}$$

Huddart et al. (2001) studied the two period Kyle model with a single informed trader ($M = 1$) and constant noise trading volatility ($\gamma = 1$). They obtain the cubic equation $8\delta^3 - 4\delta^2 - 4\delta + 1 = 0$ which coincides with (A14) if we set $M = 1$ and $\gamma = 1$. The economic solution is the largest root $\delta \approx 0.901$. Similarly, when $M > 1$, there is a unique solution that meets the second order condition and the requirement $\lambda_1 > 0$ and $\lambda_2 > 0$.

This microfoundation is used in all our numerical examples.

A.2 Proof of Theorem 1

Since arbitrageurs' prior is non-directional, we conjecture first and verify later that they do not trade in the first period, $Z_{1,n} = 0$, for $n = 1, ..., N$. Under this conjecture, we can solve their optimal strategy at $t = 2$. Arbitrageurs conjecture the market-clearing price as

$$\tilde{p}_2 = \tilde{p}_1 + \lambda_2 \left[\beta_2(\tilde{v} - \tilde{p}_1) + \sum\nolimits_{n=1}^{N} Z_{2,n}(\tilde{p}_1) + \tilde{u}_2\right]. \tag{A15}$$

They estimate $\tilde{v}$ conditional on the observed order flow y_1 and the Laplacian-Gaussian prior $\tilde{v} \sim \mathcal{LG}(\alpha, \tilde{\xi})$. For any fixed prior $\tilde{\xi} = \xi \in (0, \infty)$, the n-th trader solves her optimal strategy,

$$Z^o_{2,n}(p_1; \alpha, \xi) = \arg\max_{z_{2,n}} \mathrm{E}^{\mathcal{A}}\left[(\tilde{v} - \tilde{p}_2) z_{2,n} | p_1\right], \tag{A16}$$

where $\mathcal{A}$ denotes the prior belief $\tilde{v} \sim \mathcal{LG}(\alpha, \xi)$. We can use $Z^o_{2,-n} = \sum_{n' \neq n} Z^o_{2,n'}$ to denote the aggregate trading by all arbitrageurs except the n-th one. The first order condition is

$$\mathrm{E}^{\mathcal{A}}[\tilde{v}|p_1] - p_1 = \lambda_2 \left(\beta_2 \mathrm{E}^{\mathcal{A}}[\tilde{v}|p_1] - \beta_2 p_1 + 2z_{2,n} + \mathrm{E}^{\mathcal{A}}[Z^o_{2,-n}|p_1]\right). \tag{A17}$$

Let $\widehat{v}(p_1; \alpha, \xi) = \mathrm{E}^{\mathcal{A}}[\tilde{v}|p_1]$ be the posterior mean estimate of $\tilde{v}$. Then the strategy solution is

$$Z^o_{2,n}(p_1; \alpha, \xi) = \frac{(1 - \beta_2\lambda_2)(\widehat{v} - p_1)}{2\lambda_2} - \frac{\mathrm{E}^{\mathcal{A}}[Z^o_{2,-n}(p_1)|p_1]}{2} \tag{A18}$$

The n-th arbitrageur conjectures that every other arbitrageur solves the same problem and trades $Z^o_{2,n'} = \eta \cdot (\widehat{v} - p_1)$ for any $n' \neq n$, with a coefficient η to be determined. This suggests

$$Z^o_{2,n}(p_1; \alpha, \xi) = \frac{\widehat{v} - p_1}{2\lambda_2}\left[1 - \beta_2\lambda_2 - \eta\lambda_2(N - 1)\right]. \tag{A19}$$

As arbitrageurs make the same conjecture in a symmetric equilibrium, they can find that $\eta = \frac{1 - \beta_2\lambda_2 - \eta\lambda_2(N-1)}{2\lambda_2}$, which has a unique solution

$$\eta = \frac{1 - \beta_2\lambda_2}{\lambda_2(N + 1)} > 0. \tag{A20}$$

Since $p_1 = p_0 + \lambda_1 y_1$, their optimal strategy at $t = 2$ under the fixed prior $\mathcal{LG}(\alpha, \xi)$ is

$$Z^o_{2,n}(y_1; \alpha, \xi) = \frac{1 - \beta_2\lambda_2}{\lambda_2(N + 1)}(\widehat{v}(p_1; \alpha, \xi) - p_1) = (1 - \beta_2\lambda_2)\frac{\widehat{\theta}(y_1; \alpha, \xi)}{\lambda_2(N + 1)}, \quad \text{for} \quad n = 1, ..., N. \tag{A21}$$

Let $\tilde{x}_1 := \beta_1(\tilde{v} - p_0)$ denote the total informed order flow at $t = 1$. Then, arbitrageurs' prior about $\tilde{x}_1$ is another Laplacian-Gaussian mixture distribution:

$$f(x_1) = \frac{\alpha}{2\beta_1\xi}\exp\left(-\frac{|x_1|}{\beta_1\xi}\right) + \frac{1-\alpha}{\beta_1\sqrt{2\pi\sigma_v^2}}\exp\left(-\frac{x_1^2}{2(\beta_1\sigma_v)^2}\right). \quad \text{(A22)}$$

By Bayes' rule, the posterior probability of x_1 conditional on y_1 is found to be

$$\begin{aligned} & f(x_1|y_1) = f(y_1, x_1)/f(y_1) = f(y_1|x_1)f(x_1)/f(y_1) \\ = & \frac{\alpha/(2\beta_1\xi)}{\sqrt{2\pi\sigma_u^2}f(y_1)}\exp\left[-\frac{(y_1-x_1)^2}{2\sigma_u^2} - \frac{|x_1|}{\beta_1\xi}\right] + \frac{(1-\alpha)/\beta_1}{2\pi\sigma_u\sigma_v f(y_1)}\exp\left[-\frac{(y_1-x_1)^2}{2\sigma_u^2} - \frac{x_1^2}{2(\beta_1\sigma_v)^2}\right]. \end{aligned}$$

By direct integration, the probability density function of $\tilde{y}_1 = \beta_1(\tilde{v} - p_0) + \tilde{u}_1 = \tilde{x}_1 + \tilde{u}_1$ is

$$\begin{aligned} f(y_1) &= \frac{\alpha}{4\beta_1\xi}\exp\left(\frac{\sigma_u^2}{2(\beta_1\xi)^2}\right)\left[e^{-\frac{y_1}{\beta_1\xi}}\text{erfc}\left(\frac{\sigma_u^2/(\beta_1\xi) - y_1}{\sqrt{2}\sigma_u}\right) + e^{\frac{y_1}{\beta_1\xi}}\text{erfc}\left(\frac{\sigma_u^2/(\beta_1\xi) + y_1}{\sqrt{2}\sigma_u}\right)\right] \\ &+ \frac{1-\alpha}{\sqrt{2\pi(\sigma_u^2 + (\beta_1\sigma_v)^2)}}\exp\left[-\frac{y_1^2}{2(\sigma_u^2 + (\beta_1\sigma_v)^2)}\right]. \end{aligned} \quad \text{(A23)}$$

Define two dimensionless parameters: $\kappa(\xi) := \sigma_u/(\beta_1\xi)$ and $\varrho := (\beta_1\sigma_v/\sigma_u)^2$. By rescaling the order flows as $y_1 = y\sigma_u$, we can express $f(y_1 = y\sigma_u)$ in a dimensionless form

$$f(y) = \frac{\alpha\kappa e^{\frac{\kappa^2}{2}}}{4}\left[e^{-\kappa y}\text{erfc}\left(\frac{\kappa - y}{\sqrt{2}}\right) + e^{\kappa y}\text{erfc}\left(\frac{\kappa + y}{\sqrt{2}}\right)\right] + \frac{1-\alpha}{\sqrt{2\pi(1+\varrho)}}\exp\left[-\frac{y^2}{2(1+\varrho)}\right],$$

which is a symmetric function and decays exponentially at large $|y|$. Bayes' rule implies that

$$\mathrm{E}^{\mathcal{A}}[\tilde{x}_1 = x\sigma_u | y_1 = y\sigma_u, \xi] = \sigma_u\int_{-\infty}^{\infty} x f(x|y)dx = \sigma_u\int_{-\infty}^{\infty} x f(y|x)f(x)/f(y)dx. \quad \text{(A24)}$$

Given all the above results, we can derive the posterior expectation of $\tilde{v}$:

$$\begin{aligned} & \widehat{v}(y) = \mathrm{E}^{\mathcal{A}}[\tilde{v}|y] = \alpha\mathrm{E}^{\mathcal{A}}[\tilde{v}|y, s=1] + (1-\alpha)\mathrm{E}^{\mathcal{A}}[\tilde{v}|y, s=0] \\ = & \frac{\alpha\xi\kappa(y-\kappa)\text{erfc}\left(\frac{\kappa-y}{\sqrt{2}}\right)}{\text{erfc}\left(\frac{\kappa-y}{\sqrt{2}}\right) + e^{2\kappa y}\text{erfc}\left(\frac{\kappa+y}{\sqrt{2}}\right)} + \frac{\alpha\xi\kappa(y+\kappa)\text{erfc}\left(\frac{\kappa+y}{\sqrt{2}}\right)}{\text{erfc}\left(\frac{\kappa+y}{\sqrt{2}}\right) + e^{-2\kappa y}\text{erfc}\left(\frac{\kappa-y}{\sqrt{2}}\right)} + (1-\alpha)\lambda_1 y\sigma_u. \end{aligned} \quad \text{(A25)}$$

One can verify that $\widehat{v}(y)$ is an increasing function of $y := p_1/(\lambda_1\sigma_u)$ and its shape depends on two dimensionless parameters, α and $\kappa(\xi)$. Asymptotically, $\widehat{v}$ becomes linear in y_1:

$$\widehat{v}(y_1) \to \alpha[y_1 - \text{sign}(y_1)\kappa\sigma_u]/\beta_1 + (1-\alpha)\lambda_1 y_1, \qquad \text{as} \quad |y_1| \to \infty. \quad \text{(A26)}$$

A.3 Proof of Proposition 1

Now we examine the equilibrium existence condition. If the n-th trader does not deviate from the conjectured no-trade strategy at $t = 1$, her optimal strategy should be given by the original result $Z^o_{2,n}(p_1; \alpha, \xi) = \eta \cdot [\widehat{v}(p_1; \alpha, \xi) - p_1]$, where $p_1 = p_0 + \lambda_1 y_1 = \lambda_1(\beta_1 v + u_1)$. To verify that no arbitrageur would trade at $t = 1$, we have to examine the condition (17). Suppose the n-th trader deviates from the no-trade strategy by placing an order $Z'_{1,n}$=$z_1 \neq 0$. Then the actual total order flow at $t = 1$ will be $\tilde{y}'_1 = \beta_1\tilde{v} + \tilde{z}_1 + \tilde{u}_1$, instead of $\tilde{y}_1 = \beta_1\tilde{v} + \tilde{u}_1$ in the conjectured equilibrium. Taking as given $Z^o_{2,n'}(y'_1; \alpha, \xi) = \eta \cdot [\widehat{v}(y'_1; \alpha, \xi) - \lambda_1 y'_1]$ for any other trader $n' \neq n$, the n-th arbitrageur can solve her best response at $t = 2$ conditional on her knowledge of y_1 and z_1. It is found to be

$$\begin{aligned} Z'_{2,n}(y'_1; \alpha, \xi) &= \frac{\widehat{v}(y_1; \alpha, \xi) - \lambda_1 y'_1}{2\lambda_2} - \frac{\mathrm{E}^{\mathcal{A}}[\beta_2(\tilde{v} - \lambda_1 y'_1)|y_1, z_1] + \mathrm{E}^{\mathcal{A}}[Z^o_{2,-n}(y'_1; \alpha, \xi)|y_1, z_1]}{2} \\ &= \frac{1 - \beta_2\lambda_2}{\lambda_2(N+1)}\left([\widehat{v}(y_1; \alpha, \xi) - \lambda_1 y'_1] + \frac{N-1}{2}[\widehat{v}(y_1; \alpha, \xi) - \widehat{v}(y'_1; \alpha, \xi)]\right). \quad \text{(A27)} \end{aligned}$$

We add a few useful notations:

$$\begin{aligned} &\Delta\widehat{v} := \widehat{v}(\tilde{y}'_1; \alpha, \xi) - \widehat{v}(\tilde{y}_1; \alpha, \xi), \quad \Delta P_1 := \lambda_1(\tilde{y}'_1 - \tilde{y}_1) = \lambda_1 z_1, \quad \Delta P_2 := \tilde{p}_2(\mathbf{Z}') - \tilde{p}_2(\mathbf{Z}) \\ &\Delta Z := Z'_{2,n}(y'_1; \alpha, \xi) - Z^o_{2,n}(y_1; \alpha, \xi) = -\frac{(1 - \beta_2\lambda_2)\lambda_1 z_1}{\lambda_2(N+1)} - \frac{(1 - \beta_2\lambda_2)(N-1)}{2(N+1)\lambda_2}\Delta\widehat{v}. \end{aligned}$$

Note that $\mathbf{Z}' := [\langle 0, Z^o_{2,1}\rangle, ...\langle Z'_{1,n}, Z'_{2,n}\rangle, ...\langle 0, Z^o_{2,N}\rangle]$ differs from $\mathbf{Z} := [\langle 0, Z^o_{2,1}\rangle, ...\langle 0, Z^o_{2,N}\rangle]$ only in the n-th element $(\mathbf{Z}')_n = \langle Z'_{1,n}, Z'_{2,n}\rangle$. We can derive the following result:

$$\Delta P_2 = \lambda_1 z + \lambda_2[\Delta Z + \beta_2(\tilde{v} - \lambda_1\tilde{y}'_1) - \beta_2(\tilde{v} - \lambda_1\tilde{y}_1) + Z^o_{2,-n}(\tilde{y}'_1) - Z^o_{2,-n}(\tilde{y}_1)] = -\lambda_2\Delta Z.$$

Note that $\mathrm{E}^{\mathcal{A}}[\tilde{y}_1 z_1] = 0$ and $\mathrm{E}^{\mathcal{A}}[\widehat{v}(\tilde{y}_1; \xi) z_1] = 0$ because $\tilde{y}_1 = X_1(\tilde{v}) + \tilde{u}_1$ is symmetrically distributed and $\widehat{v}(-y_1; \xi) = -\widehat{v}(y_1; \xi)$. The extra payoff from this unilateral deviation is

$$\begin{aligned} \Delta\Pi^d_{z,n} &= \mathrm{E}^{\mathcal{A}}[(\tilde{v} - \tilde{p}_2(\mathbf{Z}'))Z'_{2,n} + (\tilde{v} - \tilde{p}_1(\mathbf{Z}'))z_1 - (\tilde{v} - \tilde{p}_2(\mathbf{Z}))Z^o_{2,n}] \\ &= -\lambda_1 z_1^2 + \mathrm{E}^{\mathcal{A}}[\mathrm{E}^{\mathcal{A}}[(\tilde{v} - \tilde{p}_2(\mathbf{Z}) + \lambda_2 Z'_{2,n})\Delta Z|\tilde{y}_1]] \\ &= -\lambda_1 z_1^2 + \mathrm{E}^{\mathcal{A}}\left[\left(\frac{\widehat{v}(\tilde{y}_1; \alpha, \xi) - \lambda_1\tilde{y}_1}{N+1} + \lambda_2\Delta Z\right) \cdot \Delta Z\right] \\ &= -\lambda_1 z_1^2 + (1 - \beta_2\lambda_2)^2\frac{\mathrm{E}^{\mathcal{A}}\left[\left(\lambda_1 z_1 + \frac{N-1}{2}\Delta\widehat{v}\right)^2\right]}{(N+1)^2\lambda_2} \\ &\quad -(1 - \beta_2\lambda_2)\frac{(N-1)\mathrm{E}^{\mathcal{A}}\left[(\widehat{v}(\tilde{y}_1; \alpha, \xi) - \lambda_1\tilde{y}_1)\Delta\widehat{v}\right]}{2(N+1)^2\lambda_2}. \quad \text{(A28)} \end{aligned}$$

It is useful to prove that $\mathrm{E}^{\mathcal{A}}\left[(\widehat{v}(\tilde{y}_1;\alpha,\xi)-\lambda_1\tilde{y}_1)(\Delta\widehat{v}-[\alpha/\beta_1+(1-\alpha)\lambda_1]z_1)\right]\geq 0$. First, we consider $z_1\geq 0$, under which $\Delta\widehat{v}(y_1,z_1)\leq\lim_{|y_1|\to\infty}\Delta\widehat{v}(y_1,z_1)=[\alpha/\beta_1+(1-\alpha)\lambda_1]z_1$ and the equality holds at infinity. This follows from Eq. (A26) and that $\widehat{v}$ is always convex for $y_1\geq 0$ and concave for $y_1\leq 0$. Second, given $\widehat{v}(-y_1;\alpha,\xi)=-\widehat{v}(y_1;\alpha,\xi)$, it follows that $\Delta\widehat{v}(y_1-\frac{z_1}{2},z_1)=\widehat{v}(y_1+\frac{z_1}{2};\alpha,\xi)-\widehat{v}(y_1-\frac{z_1}{2};\alpha,\xi)$ is an even function of y_1 and its minimum is achieved at $y_1=0$. This means $\Delta\widehat{v}(y_1,z_1)$ is shifted to the left and its minimum locates at $y_1=-\frac{z_1}{2}$. Hence, $\Delta\widehat{v}(y_1,z_1)-[\alpha/\beta_1+(1-\alpha)\lambda_1]z_1$ is non-positive and its minimum locates at $y_1=-\frac{z_1}{2}\leq 0$. Given the distributional symmetry of $\tilde{y}_1$ and the fact that $\widehat{v}(y_1;\alpha,\xi)-\lambda_1 y_1$ is an odd function, it follows that $\mathrm{E}^{\mathcal{A}}\left[(\widehat{v}(\tilde{y}_1;\alpha,\xi)-\lambda_1\tilde{y}_1)(\Delta\widehat{v}-[\alpha/\beta_1+(1-\alpha)\lambda_1]z_1)\right]\geq 0$. The same inequality holds for $z_1\leq 0$ under which $\Delta\widehat{v}(y_1,z_1)-[\alpha/\beta_1+(1-\alpha)\lambda_1]z_1\geq 0$, with its maximum located at $y_1=-\frac{z_1}{2}\geq 0$. Based on the obvious result (by symmetry) that $\mathrm{E}^{\mathcal{A}}\left[(\widehat{v}(\tilde{y}_1;\alpha,\xi)-\lambda_1\tilde{y}_1)z_1\right]=0$, the previous inequality implies that

$$\begin{aligned}\Delta\Pi^d_{z,n} &\leq -\lambda_1 z_1^2+(1-\beta_2\lambda_2)^2\frac{\mathrm{E}^{\mathcal{A}}\left[\left(\lambda_1 z_1+\frac{N-1}{2}\Delta\widehat{v}\right)^2\right]}{(N+1)^2\lambda_2}\\ &\quad -(1-\beta_2\lambda_2)\frac{(N-1)[\alpha/\beta_1+(1-\alpha)\lambda_1]}{2(N+1)^2\lambda_2}\mathrm{E}^{\mathcal{A}}\left[(\widehat{v}(\tilde{y}_1;\alpha,\xi)-\lambda_1\tilde{y}_1)z_1\right]\\ &= -\lambda_1 z_1^2+(1-\beta_2\lambda_2)^2\frac{\mathrm{E}^{\mathcal{A}}\left[(2\lambda_1 z_1+(N-1)\Delta\widehat{v})^2\right]}{4(N+1)^2\lambda_2}.\end{aligned}\tag{A29}$$

As $0\leq\Delta\widehat{v}(y_1,z_1)\leq(\alpha/\beta_1+(1-\alpha)\lambda_1)z_1$ for $z_1\geq 0$ and $(\alpha/\beta_1+(1-\alpha)\lambda_1)z_1\leq\Delta\widehat{v}(y_1,z_1)\leq 0$ for $z_1\leq 0$, the last expression of (A29) has an upper bound which is achieved when $\Delta\widehat{v}=(\alpha/\beta_1+(1-\alpha)\lambda_1)z_1$. This yields

$$\Delta\Pi^d_{z,n}(z_1;\alpha,\xi)\leq -\lambda_1 z_1^2+(1-\beta_2\lambda_2)^2\lambda_1 z_1^2\frac{\left[2+(N-1)(\alpha/(\beta_1\lambda_1)+1-\alpha)\right]^2}{4\delta(N+1)^2}\tag{A30}$$

Intuitively, the maximal benefit for the n-th trader (who unilaterally deviates) is achieved when arbitrageurs have the same extreme prior $\xi\to\infty$. In this limit, their reactions to the past order flows become the strongest and exactly linear:

$$Z^o_{2,n}(y_1;\alpha,\xi\to\infty)=\frac{\alpha(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)}{\beta_1\lambda_2}\frac{y_1}{N+1}.\tag{A31}$$

It is easy to verify that $\lim_{\xi\to\infty}\Delta\widehat{v}(\tilde{y}_1,z_1;\alpha,\xi)=\alpha z_1/\beta_1+(1-\alpha)\lambda_1 z_1$ such that

$$\lim_{\xi\to\infty}\Delta\Pi^d_{z,n}(z_1;\alpha,\xi)=-\lambda_1 z_1^2+(1-\beta_2\lambda_2)^2\lambda_1 z_1^2\frac{[2+(N-1)(\alpha/(\beta_1\lambda_1)+1-\alpha)]^2}{4\delta(N+1)^2}\tag{A32}$$

which is exactly the right-hand side of (A30). Thus, $\Delta\Pi^d_{z,n} < 0$ holds if the above coefficient in front of z_1^2 is negative. This leads to the equilibrium existence condition (23), i.e.,

$$1 + \frac{\alpha(1-\beta_1\lambda_1)}{\beta_1\lambda_1} \cdot \frac{N-1}{N+1} < \frac{2\sqrt{\lambda_2/\lambda_1}}{1-\beta_2\lambda_2}. \tag{A33}$$

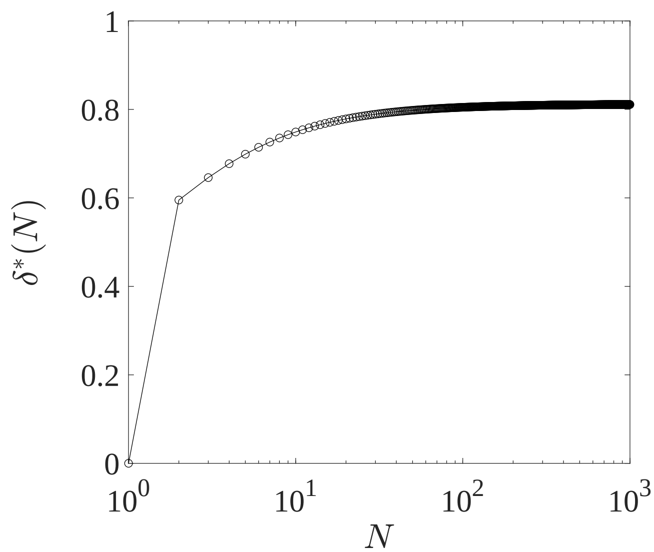

Figure 9. The critical value $\delta^*(N)$ as a function of the number of arbitrageurs N. Here, λ_t and β_t are determined in a two-period Kyle model ($M = 1$).

Consider the special case $M = 1$ in the example of microfoundation in Appendix A.1. This corresponds to a two-period a Kyle model with a single informed trader. When $\alpha = 1$, the inequality (23) lead to the condition $\delta > \delta^*(N)$ where $\delta := \lambda_2/\lambda_1$ is the ratio of Kyle lambdas and $\delta^*(N)$ is the largest root of the nonlinear equation:

$$1 + \left(\frac{N-1}{N+1}\right)\frac{2\delta}{2\delta-1} = 4\sqrt{\delta}. \tag{A34}$$

We find that $\delta^*(N{=}2) \approx 0.5951$, $\delta^*(N{=}3) \approx 0.6458$, and $\delta^*(N{=}10) \approx 0.7489$. Moreover, $\lim_{N\to\infty} \delta^*(N) \approx 0.8117$, which is the largest root of the equation: $64\delta^3 - 80\delta^2 + 24\delta - 1 = 0$. The critical values of $\delta^*(N)$ are plotted in Figure 9. The equilibrium ratio δ can vary with the ratio of noise trading volatilites (γ). In the liquidity regime $\delta > \delta^*(N)$, it is unprofitable for any arbitrageur to trade in the first period, that is, $\Delta\Pi^d_{z,n}(z_1;\xi) < 0$ for $z_1 \neq 0$. This confirms our conjecture that arbitrageurs will not trade at $t = 1$. When $\delta > \delta_\infty \approx 0.8117$, the equilibrium can host an infinite number of arbitrageurs.

A.4 Proof of Corollary 4.1

(a) follows from the property that $\widehat{v}(y_1;\alpha,\xi)$ is an odd function: $\widehat{v}(-y_1) = -\widehat{v}(y_1)$.

(b) follows from the property that $\widehat{v}(y_1;\alpha,\xi)$ is convex for $y_1 \geq 0$ and concave for $y_1 \leq 0$.

(c) follows from the result that $\lim_{|y_1|\to\infty} \widehat{v}(y_1) = (\alpha/\beta_1)[y_1 - \text{sign}(y_1)\kappa\sigma_u] + (1-\alpha)\lambda_1 y_1$.

(d) follows from the condition $\frac{\partial}{\partial y_1} Z^o_{2,n}(y_1;\alpha,\xi=\xi_c)\big|_{y_1=0} = 0$, which is equivalent to

$$1 + \kappa(\xi_c)^2 - \frac{\kappa(\xi_c)e^{-\kappa(\xi_c)^2/2}}{\text{erfc}(\kappa(\xi_c)/\sqrt{2})}\sqrt{\frac{2}{\pi}} = \lambda_1\beta_1 \tag{A35}$$

The left-hand side of (A35) is a monotonic function of κ, which decreases from 1 to 0 when κ increases from 0 to ∞. The right-hand side, $\lambda_1\beta_1$, is a constant that takes a value between 0 and 1. Hence, Eq. (A35) admits a unique positive solution $\kappa^* > 0$. Since $\kappa \coloneqq \sigma_u/(\beta_1\xi)$, we can find the unique solution (Figure 10), $\xi_c = \sigma_u/(\beta_1\kappa^*)$, which depends on σ_u and σ_v.

From the curvature property of $Z^o_{2,n}(y_1;\alpha,\xi)$, we have that $\frac{\partial^2 Z^o_{2,n}}{\partial y_1^2} > 0$ for $y_1 \geq 0$ and $\frac{\partial^2 Z^o_{2,n}}{\partial y_1^2} < 0$ for $y_1 \leq 0$. Therefore, when $\xi \geq \xi_c$, we have $\frac{\partial Z^o_{2,n}(y_1;\alpha,\xi)}{\partial y_1} \geq 0$, showing that $Z^o_{2,n}(y_1;\alpha,\xi)$ is an increasing function of y_1 which has only one root at $y_1 = 0$. In contrast, $Z^o_{2,n}(y_1;\alpha,\xi)$ becomes a non-monotonic function when $\xi < \xi_c$ and has three different roots.

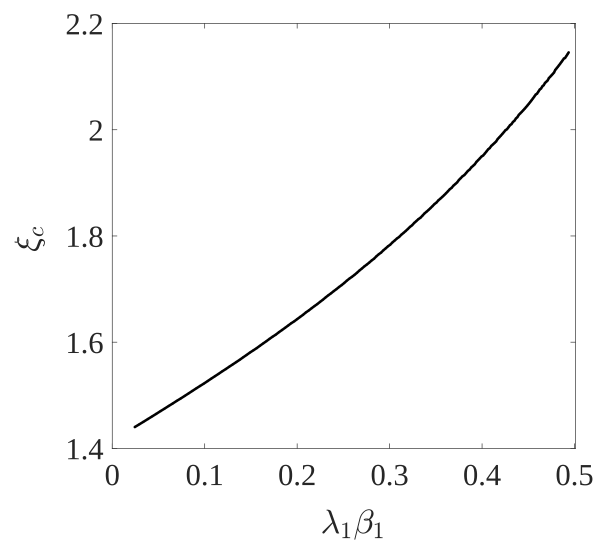

Figure 10. The critical value ξ_c as a function of the value of $\lambda_1\beta_1$, where λ_1 and β_1 are determined in a two-period Kyle model.

A.5 Proof of Corollary 4.2

First, when $\xi_L < \xi_c \leq \xi_H$, arbitrageurs' best response at $t = 2$ is to stop trading on any $y_1 \in [-K_L, K_L]$ where K_L is the positive root of the equation $Z^o_{2,n}(y_1; \alpha, \xi_L) = 0$. If other arbitrageurs do not trade on $y_1 \in [-K_L, K_L]$, any individual arbitrageur would not deviate because buying or selling this asset could lose money in case when the true prior turns out to be on the opposite side of such trading. For any $|y_1| > K_L$, the max-min strategy simply follows $Z^o_{2,n}(y_1; \alpha, \xi_L)$. This is because any unilateral deviation from this most conservative strategy may lose money in case that the reality happens to be the lowest prior ξ_L. Thus, no one would trade more than the most conservative strategy $Z^o_{2,n}(y_1; \alpha, \xi_L)\mathbf{1}_{|y_1|>K_L}$.

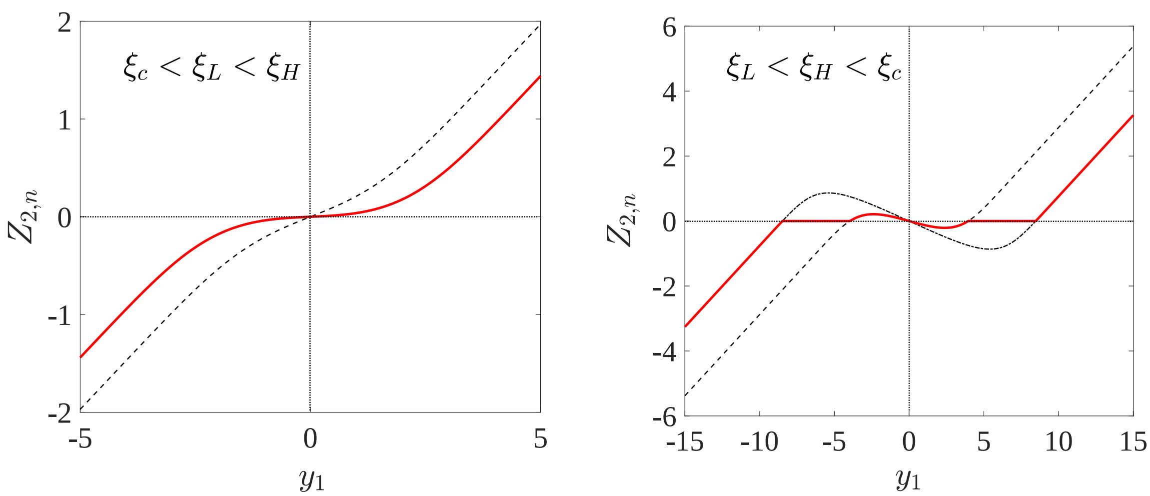

Figure 11. The equilibrium max-min strategies when $\xi_c < \xi_L < \xi_H$ and $\xi_L < \xi_H < \xi_c$.

Second, when $\xi_c \leq \xi_L < \xi_H$, the two extreme strategies $Z^o_{2,n}(y_1; \alpha, \xi_H)$ and $Z^o_{2,n}(y_1; \alpha, \xi_L)$ agree on the trading direction for all realized y_1. Thus, the max-min strategy is $Z^o_{2,n}(y_1; \alpha, \xi_L)$, as shown by the red solid line in the left panel of Figure 11.

Last, when $\xi_L < \xi_H < \xi_c$, both $Z^o_{2,n}(y_1; \alpha, \xi_L)$ and $Z^o_{2,n}(y_1; \alpha, \xi_H)$ are non-monotonic, and each has three roots. The max-min strategy is $Z^o_{2,n}(y_1; \alpha, \xi_H)\mathbf{1}_{|y_1|<K_H} + Z^o_{2,n}(y_1; \alpha, \xi_L)\mathbf{1}_{|y_1|>K_L}$, where K_H represents the positive root of the equation $Z^o_{2,n}(y_1; \alpha, \xi_H) = 0$. Since $K_H < K_L$, this strategy has two no-trade zones, $[-K_L, -K_H]$ and $[K_H, K_L]$, and three trading zones, $[-K_H, K_H]$, $[-\infty, -K_L]$, and $[K_L, \infty]$. The solution is shown by the red solid line in the right panel of Figure 11. It is too complicated to be practically relevant.

Note that K_L and $K(\xi_L)$ are very close but not identical. K_L is the positive point where $Z_{2,n}(y_1; \alpha, K(\xi_L))$ crosses the horizontal y_1-axis, whereas $K(\xi_L)$ is the horizontal intercept of the asymptote of $Z_{2,n}(y_1; \alpha, K(\xi_L))$. The difference between K_H and $K(\xi_H)$ is similar.

A.6 Proof of Theorem 2

First, we verify that, under (C1), (C2), and (C3), each arbitrageur will not deviate from the strategy (28) at $t = 2$. By symmetry, it suffices to consider the positive domain.

For any realized order flow $y_1 \in [0, K(\xi_w)]$, the n-th arbitrageur will not deviate to buy any share of this asset, because choosing $Z'_{2,n} > 0$ may lose money under the lowest prior ξ_L:

$$\begin{aligned} \mathrm{E}^{\mathcal{A}}[\Delta\tilde{\pi}_{z,n}|y_1, \tilde{\xi} = \xi_L] &= \mathrm{E}^{\mathcal{A}}\left[\left(\tilde{v} - \lambda_1 y_1 - \lambda_2[\beta_2(\tilde{v} - \lambda_1 y_1) + Z'_{2,n} + \tilde{u}_2]\right) Z'_{2,n}\big|y_1, \tilde{\xi} = \xi_L\right] - 0 \\ &= (1 - \beta_2\lambda_2)\widehat{\theta}(y_1; \alpha, \xi_L)Z'_{2,n} - \lambda_2 Z'^2_{2,n} < 0. \end{aligned} \tag{A36}$$

The inequality is due to $(1-\beta_2\lambda_2)\widehat{\theta}(y_1; \alpha, \xi_L)=\lambda_2(N+1)Z_{2,n}(y_1; \alpha, \xi_L) < 0$ for $y_1 \in [0, K(\xi_w)]$, which is implied by (C1) $K(\xi_w) < K_L$, (C3) $\xi_L < \xi_c$, and Corollary 4.1(d). Similarly, each arbitrageur would not deviate by choosing any $Z'_{2,n} < 0$ as it may lose money under ξ_H:

$$\mathrm{E}^{\mathcal{A}}[\Delta\tilde{\pi}_{z,n}|y_1, \tilde{\xi} = \xi_H] = (1 - \beta_2\lambda_2)\widehat{\theta}(y_1; \alpha, \xi_H)Z'_{2,n} - \lambda_2 Z'^2_{2,n} < 0. \tag{A37}$$

The inequality is due to $(1 - \beta_2\lambda_2)\widehat{\theta}(y_1; \alpha, \xi_H)=\lambda_2(N + 1)Z_{2,n}(y_1; \alpha, \xi_H) > 0$ since $\xi_H > \xi_c$. By the max-min criterion, arbitrageurs will not deviate from no trading for $y_1 \in [0, K(\xi_w)]$.

For $y_1 \in (K(\xi_w), \infty)$, each arbitrageur will not trade less than the amount of $Z^\infty(y_1; \alpha, \xi_w)$, because doing this would violate either (C1) or (C2) or both. Each arbitrageur will not trade more that $Z^\infty(y_1; \alpha, \xi_w)$ either, because choosing any $Z'_{2,n}(y_1) > Z^\infty(y_1; \alpha, \xi_w)$ may lose more or earn less than the buying decision made along $Z^\infty(y_1; \alpha, \xi_w)$. Define $Z_\Delta \coloneqq Z'_{2,n}(y_1) - Z^\infty(y_1; \alpha, \xi_w)$. The difference of payoffs from this unilateral deviation under ξ_L is

$$\begin{aligned} \mathrm{E}^{\mathcal{A}}[\Delta\tilde{\pi}_{z,n}|y_1, \xi_L] &= \mathrm{E}^{\mathcal{A}}\left[\left(\tilde{v} - \lambda_1 y_1 - \lambda_2[\beta_2(\tilde{v} - \lambda_1 y_1) + Z'_{2,n} + Z_{2,-n} + \tilde{u}_2]\right) Z'_{2,n}\big|y_1, \xi_L\right] \\ &\quad -\mathrm{E}^{\mathcal{A}}\left[(\tilde{v} - \lambda_1 y_1 - \lambda_2[\beta_2(\tilde{v} - \lambda_1 y_1) + Z^\infty + Z_{2,-n} + \tilde{u}_2]) Z^\infty\big|y_1, \xi_L\right] \\ &= \mathrm{E}^{\mathcal{A}}\left[Z_\Delta[(1 - \beta_2\lambda_2)\tilde{\theta} - \lambda_2(Z^\infty + Z_{2,-n})]\big|y_1, \xi_L\right] - \lambda_2 Z'_{2,n}Z_\Delta. \end{aligned} \tag{A38}$$

Let $Z_L \coloneqq (1 - \beta_2\lambda_2)\frac{\widehat{\theta}(y_1;\alpha,\xi_L)}{\lambda_2(N+1)}$. Since everyone else follows Z^∞ and $Z_L < Z^\infty < Z'_{2,n}$, we have

$$\begin{aligned} \mathrm{E}^{\mathcal{A}}[\Delta\tilde{\pi}_{z,n}|y_1, \xi_L] &= \lambda_2 Z_\Delta[(N + 1)Z_L - Z^\infty - (N - 1)Z^\infty] - \lambda_2 Z'_{2,n}Z_\Delta \\ &= \lambda_2 Z_\Delta[(N + 1)Z_L - NZ^\infty - Z'_{2,n}] < 0. \end{aligned} \tag{A39}$$

Thus, each arbitrageur will only trade $Z^\infty(y_1; \alpha, \xi_w)$ when $y_1 > K(\xi_w)$. By symmetry, the full equilibrium strategy is $Z^\infty(y_1; \alpha, \xi_w)\mathbf{1}_{|y_1|>K(\xi_w)}$. No one will deviate from it at $t = 2$.

The equilibrium condition (C1) is central to Theorem 2. This is endogenously implied by the existence of a debiased equilibrium. If (C1) is absent, (C3) alone is sufficient to direct the economy to the equilibrium in Corollary 4.2 and the presence of (C2) makes no difference. When (C1) holds, (C2) plays an important role in regularizing the robust optimization problem. Without (C2), the other conditions (C1) and (C3) cannot support the equilibrium in Theorem 2. This is because if candidate strategies were allowed to be non-convex in the positive domain of y_1 or non-concave in the negative domain, then for any strategy that satisfies (C1), arbitrageurs would always find some deviations that trade more conservatively than that strategy. Such deviations are permitted by the gap between $Z^{\infty}(y_1;\alpha,\xi_w)$ and $Z^{\infty}(y_1;\alpha,\xi_L)$, which is an implication of (C1).

Under (C1) and (C2), all admissible strategies must be within the shaded area in Figure 2, enclosed by $Z^o_{2,n}(y_1;\alpha,\xi\to 0)$, $Z^o_{2,n}(y_1;\alpha,\xi\to\infty)$, and $Z^{\infty}(y_1;\alpha,\xi_w)$. This can be understood by just looking at the positive domain of y_1. It is apparently irrational for any strategy to go above $Z^o_{2,n}(y_1;\alpha,\xi\to\infty)$ or below $Z^o_{2,n}(y_1;\alpha,\xi\to 0)$. Moreover, (C2) means that the first derivative of any admissible strategy is non-decreasing in the positive domain of y_1. Such a strategy cannot cross the asymptote $Z^{\infty}(y_1;\alpha,\xi_w)$ without violating (C1).

It remains to verify that no arbitrageur would find it utility-improving to trade at $t=1$, given that other arbitrageurs only trade at $t=2$ using the strategy (28). The proof here is similar to A.2 for Theorem 1. Intuitively, each arbitrageur would not trade at $t=1$ since it would risk trading on the opposite side of the true fat-tail signal. But it might be profitable if other arbitrageurs were overly misled by the secret "disruptive" trading. Suppose the n-th arbitrageur is an instigator who considers to trade $Z'_{1,n}=z_1\neq 0$, in an effort to confuse other arbitrageurs (now momentum traders). To save notations, we will use K to represent $K(\xi_w)$. Given her market power and unilateral deviation, the instigator understands the composition of order flows: $\tilde{y}'_1=\beta_1(\tilde{v}-p_0)+z_1+\tilde{u}_1$ and $\tilde{y}'_2=\beta_2(\tilde{v}-\lambda_1\tilde{y}'_1)+Z'_{2,n}(\tilde{y}_1,z_1)+Z_{2,-n}(\tilde{y}'_1;\alpha,K)+\tilde{u}_2$. Here, $Z_{2,-n}\coloneqq\sum_{n'\neq n}Z_{2,n'}(y'_1;\alpha,K)$ is the total order flow placed by other arbitrageurs who will estimate $\tilde{\theta}=\tilde{v}-\lambda_1 y'_1$ based on the observed y'_1 without knowing that y'_1 contains an uninformed order flow z_1 from the instigator. Of course, the instigator's estimate of $\tilde{\theta}$ is correctly based on $y_1=\beta_1(\tilde{v}-p_0)+u_1$ instead of y'_1, because she knows the order z_1 placed by herself. Being averse to the model risk, the instigator has the objective function:

$$\max_{z_2}\min_{\xi}\mathrm{E}^{\mathcal{A}}[(\tilde{v}-\lambda_1\tilde{y}'_1-\lambda_2\tilde{y}'_2)z_2|y_1,z_1,\alpha,\xi], \tag{A40}$$

The instigator can exploit the possibility that other arbitrageurs are (unknowingly) biased by her uninformed trade z_1 since they follow the strategy $Z_{2,n'}(y'_1=y_1+z_1;\alpha,K)$ for each $n'\neq n$. For a given prior $\xi\in[\xi_L,\xi_H]$, the instigator's optimal strategy at $t=2$ is

$$Z^o_{2,n}(y_1,z_1;\alpha,\xi)=(1-\beta_2\lambda_2)\frac{\mathrm{E}^{\mathcal{A}}[\tilde{v}|y_1,\alpha,\xi]-\lambda_1(y_1+z_1)}{2\lambda_2}-\frac{1}{2}\sum_{n'\neq n}Z_{2,n'}(y_1+z_1;\alpha,K). \tag{A41}$$

Since the last term is independent of her prior ξ, it will remain in her max-min strategy:

$$Z'_{2,n}(y_1, z_1) = \arg\max_{z_2} \min_{\xi} \mathrm{E}^{\mathcal{A}} \left[(\tilde{v} - \lambda_1 y'_1 - \lambda_2(y'_1 + z_2))z_2 | y_1, z_1, \alpha, \xi\right] - \frac{1}{2} \sum_{n' \neq n} Z_{2,n'}(y'_1; \alpha, K).$$

The first term is actually the problem we solved earlier as if there was only one arbitrageur and the first-period order flow was y'_1 instead of y_1. Nonetheless, the instigator's estimate of $\tilde{v}$ is correctly based on y_1 (not y'_1). By deducting her own price impact $\lambda_1 z_1$ from this problem, she can find that the max-min solution to the remaining problem is simply

$$Z_{2,n}(y_1; \alpha, K)\big|_{N=1} = \frac{\alpha(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)}{2\beta_1\lambda_2} \left[y_1 - \mathrm{sign}(y_1)K\right] \mathbf{1}_{|y_1|>K} = \frac{N+1}{2} Z_{2,n}(y_1; \alpha, K).$$

Taking into account all those results, she will find her max-min strategy given by

$$\begin{aligned} Z'_{2,n}(y_1, z_1) &= \frac{(N+1)}{2} Z_{2,n}(y_1; \alpha, K) - (1-\beta_2\lambda_2)\frac{\lambda_1 z_1}{2\lambda_2} - \frac{N-1}{2} Z_{2,n}(y'_1; \alpha, K) \\ &= Z_{2,n}(y_1; \alpha, K) - (1-\beta_2\lambda_2)\frac{z_1}{2\delta} - \frac{\alpha(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)(N-1)}{2\beta_1\lambda_2(N+1)} D(y_1, z_1), \end{aligned} \tag{A42}$$

where we define the difference between two soft-thresholding functions:

$$D(y_1, z_1) \coloneqq [y_1 + z_1 - \mathrm{sign}(y_1 + z_1)K]\mathbf{1}_{|y_1+z_1|>K} - [y_1 - \mathrm{sign}(y_1)K]\mathbf{1}_{|y_1|>K}. \tag{A43}$$

Ex ante, the expected trading profit of this instigator (the n-th arbitrageur) is

$$\Pi^d_{z,n}(z_1) = \mathrm{E}^{\mathcal{A}}[(\tilde{v} - \lambda_1\tilde{y}'_1)z_1 + (\tilde{v} - \lambda_1\tilde{y}'_1 - \lambda_2\tilde{y}'_2) \cdot Z'_{2,n}(\tilde{y}_1, z_1)], \tag{A44}$$

and hence the extra profit attributable to her unilateral deviation $\langle Z'_{1,n}, Z'_{2,n}\rangle$ is

$$\Delta\Pi^d_{z,n}(z_1) = \Pi^d_{z,n}(z_1) - \mathrm{E}^{\mathcal{A}}[(\tilde{v} - \lambda_1\tilde{y}_1 - \lambda_2\tilde{y}_2) \cdot Z_{2,n}(\tilde{y}_1; \alpha, K)], \tag{A45}$$

where $\tilde{y}_1 = \beta_1(\tilde{v} - p_0) + \tilde{u}_1$ and $\tilde{y}_2 = \beta_2(\tilde{v} - \lambda_1\tilde{y}_1) + \sum_{n=1}^N Z_{2,n}(\tilde{y}_1; \alpha, K) + \tilde{u}_2$. We can derive

$$\begin{aligned} \Delta\Pi^d_{z,n} &= -\lambda_1 z_1^2 + \lambda_2 \mathrm{E}^{\mathcal{A}}[(Z'_{2,n}(\tilde{y}_1, z_1))^2] - \lambda_2 \mathrm{E}^{\mathcal{A}}[(Z_{2,n}(\tilde{y}_1; \alpha, K))^2] \\ &= -\lambda_1 z_1^2 + \lambda_2 \mathrm{E}^{\mathcal{A}}\Bigg[\left((1-\beta_2\lambda_2)\frac{z_1}{2\delta} + \frac{\alpha(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)(N-1)}{2\beta_1\lambda_2(N+1)} D\right)^2 \\ &\quad - \frac{\alpha(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)(N-1)}{\beta_1\lambda_2(N+1)} Z_{2,n} D\Bigg]. \end{aligned} \tag{A46}$$

By symmetry, $\Delta\Pi^d_{z,n}(-z_1) = \Delta\Pi^d_{z,n}(z_1)$. We can simply focus on the case of $z_1 > 0$. It takes some straightforward but tedious calculation to arrive at the following result (when $z_1 > 0$):

$$Z_{2,n}(y_1;\alpha,K)\cdot(D-z_1) = \begin{cases} \max\{A(y_1,z_1),0\} & \text{if } z_1 \le 2K \\ \max\{A(y_1,z_1)\mathbf{1}_{y_1<K-z_1} + B(y_1,z_1)\mathbf{1}_{y_1\ge K-z_1},0\} & \text{if } z_1 > 2K \end{cases}$$

where $A(y_1,z_1) \coloneqq -\frac{\alpha(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)}{\beta_1\lambda_2(N+1)}(y_1+K)(y_1+z_1+K)$ is a quadratic function of y_1 for any given z_1 and $B(y_1,z_1) \coloneqq A(K-z_1,z_1)\frac{y_1+K}{2K-z_1}$ is a linear function of y_1 for any given z_1. The above result proves that $Z_{2,n}(\tilde{y}_1;\alpha,K)D(\tilde{y}_1,z_1) \ge Z_{2,n}(\tilde{y}_1;\alpha,K)z_1$, which further implies

$$\begin{aligned} \Delta\Pi^d_{z,n} &\le -\lambda_1 z_1^2 + \lambda_2 \mathrm{E}^{\mathcal{A}}\left[\left((1-\beta_2\lambda_2)\frac{z_1}{2\delta} + \frac{\alpha(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)(N-1)}{2\beta_1\lambda_2(N+1)}D\right)^2\right. \\ &\le -\lambda_1 z_1^2 + \lambda_2\left[\frac{1-\beta_2\lambda_2}{2\delta} + \frac{\alpha(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)(N-1)}{2\beta_1\lambda_2(N+1)}\right]^2 z_1^2. \end{aligned} \tag{A47}$$

The second step is by the property that $0 \le D \le z_1$ if $z_1 \ge 0$ and $z_1 \le D \le 0$ if $z_1 \le 0$ and $\mathrm{E}^{\mathcal{A}}[Z_{2,n}]z_1 = 0$. The equality in (A47) holds when $\xi_w \to \infty$ so that $K(\xi_w) \to 0$ and $D \to z_1$.

It is not a profitable deviation (i.e., $\Delta\Pi^d_{z,n} < 0$) if the coefficient of z_1^2 in (A47) is negative. This coefficient condition leads to the same equilibrium condition (23) in Proposition 1:

$$1 + \frac{\alpha(1-\beta_1\lambda_1)}{\beta_1\lambda_1}\cdot\frac{N-1}{N+1} < \frac{2\sqrt{\lambda_2/\lambda_1}}{1-\beta_2\lambda_2}, \tag{A48}$$

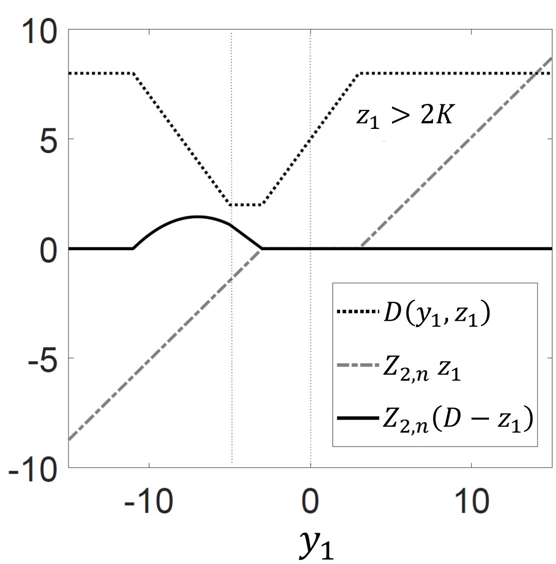

Figure 12. $D(y_1,z_1)$, $Z_{2,n}(y_1;\alpha,K)z_1$, and $Z_{2,n}(y_1;\alpha,K)\cdot(D-z_1)$ when $z_1 > 2K$.

A.7 Proof of Theorem 3

By definition, if a strategy can be written as a soft-thresholding function, then we may map it into a LASSO strategy which involves some LASSO estimates of economic variables. For $\alpha > 0$, the robust strategy (28) in Theorem 2 is a soft-thresholding function of y_1:

$$Z_{2,n}(y_1; \alpha, K(\xi_w)) = Z^{\infty}(y_1; \alpha, \xi_w)\mathbf{1}_{|y_1|>K(\xi_w)} = \frac{\alpha(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)}{\beta_1\lambda_2(N+1)}\mathcal{S}(y_1; K(\xi_w)). \quad \text{(A49)}$$

Similar to the optimal strategy (19), the robust strategy (A49) can be written as

$$Z_{2,n}(y_1; \alpha, K(\xi_w)) = \frac{\alpha(1-\beta_2\lambda_2)}{\lambda_2(N+1)} \cdot \widehat{\theta}^{\text{imp}}(y_1; \xi_w), \quad \text{(A50)}$$

where $\widehat{\theta}^{\text{imp}} = \frac{1-\beta_1\lambda_1}{\beta_1}\mathcal{S}(y_1; K(\xi_w))$ is the strategy-implied estimator. It remains to show that $\widehat{\theta}^{\text{imp}}$ corresponds to some LASSO estimate of $\tilde{\theta}$. If we define

$$\widehat{\theta}^{\text{lasso}}(y_1; \xi_w) \coloneqq \arg\min_{\theta}\left\{\frac{1}{2}\left|y_1 - \frac{\beta_1\theta}{1-\beta_1\lambda_1}\right|^2 + \frac{\sigma_u^2|\theta|}{(1-\beta_1\lambda_1)^2\xi_w}\right\}, \quad \text{(A51)}$$

then by Equations (2), (3), and (5), we can show that

$$\widehat{\theta}^{\text{lasso}} = \frac{1-\beta_1\lambda_1}{\beta_1}[y_1 - \text{sign}(y_1)K(\xi_w)]\mathbf{1}_{|y_1|>K(\xi_w)} = \frac{1-\beta_1\lambda_1}{\beta_1}\mathcal{S}(y_1; K(\xi_w)) = \widehat{\theta}^{\text{imp}}. \quad \text{(A52)}$$

This proves that the robust strategy (28) is a LASSO strategy since we have

$$Z_{2,n}(y_1; \alpha, K(\xi_w)) = \frac{\alpha(1-\beta_2\lambda_2)}{\lambda_2(N+1)} \cdot \widehat{\theta}^{\text{lasso}}(y_1; \xi_w). \quad \text{(A53)}$$

We can further define a LASSO estimate of $\tilde{v}$ as

$$\widehat{v}^{\text{lasso}}(y_1; \xi_w) \coloneqq \arg\min_{v}\left\{\frac{1}{2}|y_1 - \beta_1 v|^2 + \frac{\sigma_u^2}{\xi_w}|v|\right\} = \frac{1}{\beta_1}\mathcal{S}(y_1; \kappa\sigma_u). \quad \text{(A54)}$$

Since $\kappa_a\sigma_u < K(\xi_w) = \frac{\kappa_a\sigma_u}{1-\beta_1\lambda_1}$, it follows that $\widehat{\theta}^{\text{lasso}} = (\widehat{v}^{\text{lasso}} - \lambda_1 y_1)\mathbf{1}_{|y_1|>K(\xi_w)}$ and thus

$$Z_{2,n}(y_1; \alpha, K(\xi_w)) = \frac{\alpha(1-\beta_2\lambda_2)(\widehat{v}^{\text{lasso}} - \lambda_1 y_1)\mathbf{1}_{|y_1|>K(\xi_w)}}{\lambda_2(N+1)}. \quad \text{(A55)}$$

Eq. (A53) and Eq. (A55) together prove Eq. (29) in Theorem 3.

A.8 Proof of Proposition 2

When $\alpha = 1$, a fixed Laplacian-Gaussian mixture prior $\mathcal{LG}(\alpha, \xi)$ reduces to a pure Laplace prior $\mathcal{L}(0, \xi)$. Conditional on this prior and the order flow $y_1 = \beta_1(\tilde{v} - p_0) + \tilde{u}_1$ with $p_0 = 0$, the posterior distribution $f(v|y_1)$ is equal to $f(y_1|v)f(v)/f(y_1)$ by Bayes' rule and hence the *maximum a posteriori* (MAP) estimate of $\tilde{v}$ is given by

$$\widehat{v}^{\text{map}} = \arg\max_{v} \ \exp\left[-\frac{(y_1 - \beta_1 v)^2}{2\sigma_u^2} - \frac{|v|}{\xi}\right] = \arg\min_{v} \left\{\frac{|y_1 - \beta_1 v|^2}{2} + \frac{\sigma_u^2}{\xi}|v|\right\}. \quad \text{(A56)}$$

When $\xi = \xi_w$, Eq. (A56) becomes the same LASSO objective function (30) that defines $\widehat{v}^{\text{lasso}}$. The first order condition of (A56) is $y_1(v) = \beta_1 v + \text{sign}(v)\kappa(\xi)\sigma_u$, where $\kappa(\xi) := \sigma_u/(\beta_1\xi)$. Inverting this function $y_1(v)$ yields the MAP estimator which has a learning threshold $\kappa\sigma_u$. When $\xi = \xi_w$, it coincides with the soft-thresholding expression of $\widehat{v}^{\text{lasso}}$ in Eq. (30):

$$\widehat{v}^{\text{map}}(y_1; \alpha = 1, \xi_w) = \frac{1}{\beta_1}\left[y_1 - \text{sign}(y_1)\kappa\sigma_u\right]\mathbf{1}_{|y_1| > \kappa\sigma_u} = \frac{1}{\beta_1}\mathcal{S}(y_1; \kappa\sigma_u) = \widehat{v}^{\text{lasso}}(y_1; \xi_w). \quad \text{(A57)}$$

(A56) and (A57) demonstrate the statistical interpretation of LASSO by Tibshirani (1996). Figure 13 plots the posterior distribution $f(v|y_1; \alpha = 1, \xi)$ for three values of y_1 with $\sigma_u = 1$. It illustrates the effect of a sharply peaked Laplace prior on suppressing nonzero estimates. When y_1 is at or below the learning threshold $\kappa\sigma_u$, the posterior remains sharply peaked at the origin such that $\widehat{v}^{\text{map}} = 0$. When y_1 exceeds the threshold $\kappa\sigma_u$, the posterior mode shifts to the right such that $\widehat{v}^{\text{map}} > 0$. The l_1 penalty term in LASSO has the same effect.

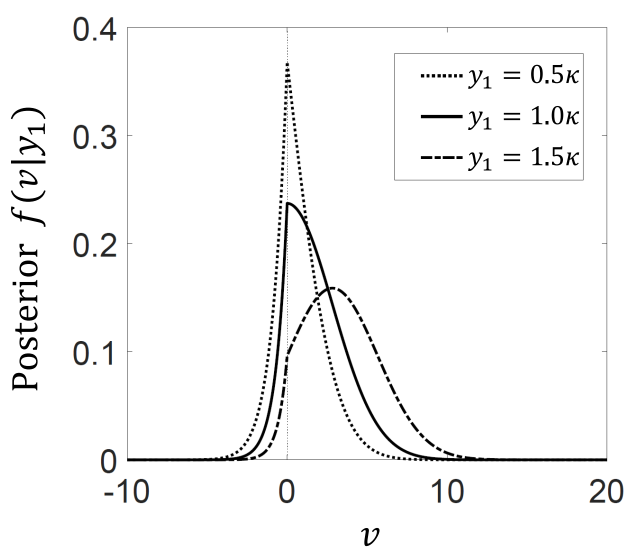

Figure 13. The posterior distributions $f(v|y_1; \alpha = 1, \xi)$ for $y_1 = 0.5\kappa$, $y_1 = \kappa$, and $y_1 = 1.5\kappa$.

Now consider an otherwise identical economy where arbitrageurs naively use the MAP rule to estimate $\tilde{v}$ based on a pure Laplace prior $\tilde{v} \sim \mathcal{L}(0, \xi_w)$. We can easily show that their heuristic feedback trading strategy coincides with the robust LASSO strategy (28):

$$\begin{aligned} & Z_{2,n}^{\text{map}}(y_1; \alpha = 1, \xi_w) = \frac{(1-\beta_2\lambda_2)}{\lambda_2(N+1)} [\widehat{v}^{\text{map}}(y_1; \alpha = 1, \xi_w) - \lambda_1 y_1] \mathbf{1}_{|y_1| > K(\xi_w)} \\ = \; & Z_{2,n}(y_1; \alpha = 1, K(\xi_w)) = \frac{(1-\beta_1\lambda_1)(1-\beta_2\lambda_2)}{\beta_1\lambda_2(N+1)} [y_1 - \text{sign}(y_1) K(\xi_w)] \, \mathbf{1}_{|y_1| > K(\xi_w)}, \end{aligned} \quad \text{(A58)}$$

where the term $\mathbf{1}_{|y_1| > K(\xi_w)}$ is imposed to guarantee the practice of positive feedback trading. Eq. (A58) demonstrates the observational equivalence between the heuristic MAP strategy $Z_{2,n}^{\text{map}}(y_1; \alpha = 1, \xi_w)$ and the robust LASSO strategy $Z_{2,n}(y_1; \alpha = 1, \xi_w)$.

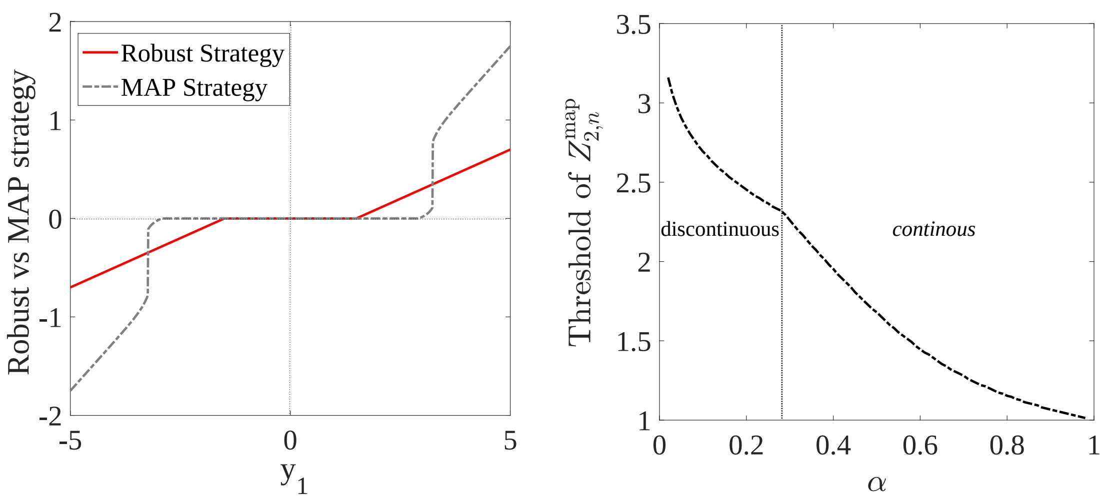

Figure 14. Left: the LASSO strategy $Z_{2,n}(y_1)$ and the MAP strategy $Z_{2,n}^{\text{map}}(y_1)$ when $\alpha = 0.5$. Right: the trading threshold of MAP strategy versus α.

For any $\alpha \in (0, 1)$, this observational equivalence fails because the MAP estimate becomes

$$\widehat{v}^{\text{map}} := \arg\max_v \left[\frac{\alpha}{2\xi} \exp\left(-\frac{(y_1 - \beta_1 v)^2}{2\sigma_u^2} - \frac{|v|}{\xi} \right) + \frac{1-\alpha}{\sqrt{2\pi\sigma_v^2}} \exp\left(-\frac{(y_1 - \beta_1 v)^2}{2\sigma_u^2} - \frac{v^2}{2\sigma_v^2} \right) \right].$$

which is always different from the LASSO estimate $\widehat{v}^{\text{lasso}}$. There is no analytical solution to this problem, but we can numerically determine the MAP estimator and the corresponding feedback trading strategy. A numerical example is shown in Figure 14 (left). When $\alpha \in (0, 1)$, the heuristic MAP strategy has a trading threshold always larger than that of the robust LASSO strategy; see the right panel of Figure 14. We find that when α is sufficiently small, the MAP strategy $Z_{2,n}^{\text{map}}(y_1; \alpha, \xi_w)$ becomes discontinuous at its trading thresholds. This can also be seen from Figure 3 in the main text. Unlike the robust LASSO strategy which scales linearly with α, the MAP strategy has the same asymptotes as they are independent of α. All the above results show the significant discrepancy between these two strategies.

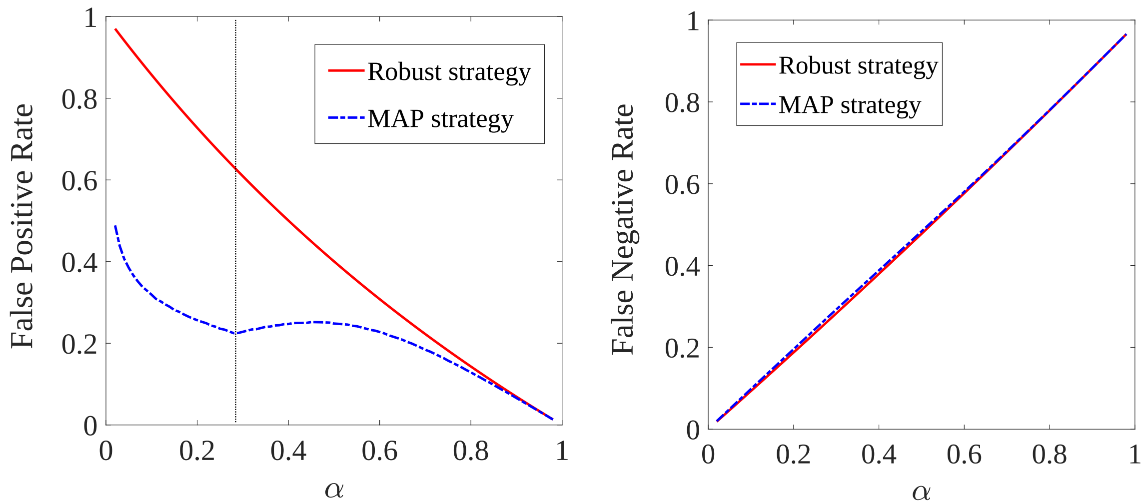

Figure 15. Left: false positive rates for the robust LASSO strategy $Z_{2,n}$ and the heuristic MAP strategy $Z_{2,n}^{\text{map}}$. Right: false positive rates for both types of strategies.

Next we compare the performances of the two strategies over the entire range of $\alpha \in (0, 1)$. The MAP strategy is not Bayesian rational as it tends to produce the *all-or-none* responses: whenever the order flow y_1 exceeds its trading thresholds, the MAP strategy treats y_1 as if it contains a Laplacian signal for sure and thus commits to the maximal trading intensity. This feature of binary classification stems from the heuristic MAP rule as it simply picks up the posterior mode and brutally ignores all the remaining posterior information. When the frequency of fat-tail shocks declines (i.e., as α decreases), the MAP strategy uses a larger threshold to achieve its binary classification. In contrast, the robust LASSO strategy does not alter the threshold since $K(\xi_w)$ is independent of α. Of course, a wider inaction zone implies a lower rate of Type I errors (false positives), as shown in Figure 15. In terms of Type II errors (false negatives), the two strategies are almost identical in performance.

The false positive rate can only reflect the accuracy in the direction of responses, not in the magnitude of responses. The MAP strategy may overly trade given its all-or-none feature. Figure 4 shows the expected total trading profits at $\alpha = 0.5$ under three types of strategies: the robust LASSO strategy $Z_{2,n}(y_1; \alpha, K(\xi_w))$, the optimal strategy $Z_{2,n}^o(y_1; \alpha, \xi_w)$, and the heuristic MAP strategy $Z_{2,n}^{\text{map}}(y_1; \alpha, \xi_w)$. One can see that for a wide range of ξ_w (relative to the true prior ξ_v), the LASSO strategy is much more profitable than the MAP strategy. The MAP strategy can easily lose money when it has estimate bias or faces intense competition.

A.9 Proof of Theorem 4

In the symmetric equilibrium of Theorem 1 where arbitrageurs follow the same optimal strategy $Z^o_{2,n}(\tilde{y}_1;\alpha,\xi_w)$, their expected total profits under the physical measure is given by

$$
\begin{aligned}
& \mathrm{E}\left[\sum_{n=1}^{N}(\tilde{v}-\tilde{p}_2)Z^o_{2,n}(\tilde{y}_1;\alpha,\xi_w)\right] \\
= \;& \mathrm{E}\left[N\left(\tilde{v}-\lambda_1\tilde{y}_1-\lambda_2\beta_2(\tilde{v}-\lambda_1\tilde{y}_1)-\sum_{n=1}^{N}\lambda_2 Z^o_{2,n}\right)Z^o_{2,n}\right] \\
= \;& (1-\beta_2\lambda_2)^2\mathrm{E}\left[\frac{N(N+1)\tilde{\theta}-N^2\widehat{\theta}(\tilde{y}_1;\alpha,\xi_w)}{(N+1)}\cdot\frac{\widehat{\theta}(\tilde{y}_1;\alpha,\xi_w)}{(N+1)\lambda_2}\right] \\
= \;& (1-\beta_2\lambda_2)^2\frac{N\mathrm{E}\left[\widehat{\theta}(\tilde{y}_1;\alpha,\xi_v)\widehat{\theta}(\tilde{y}_1;\alpha,\xi_w)\right]}{(N+1)\lambda_2}-(1-\beta_2\lambda_2)^2\frac{N^2\mathrm{E}\left[\widehat{\theta}(\tilde{y}_1;\alpha,\xi_w)^2\right]}{(N+1)^2\lambda_2}, \quad \text{(A59)}
\end{aligned}
$$

where we have used Eq. (19) as well as the following property

$$
\mathrm{E}\left[\tilde{\theta}\cdot\widehat{\theta}(\tilde{y}_1;\alpha,\xi_w)\right]=\mathrm{E}\left[\mathrm{E}\left[\tilde{\theta}\middle|\tilde{y}_1;\alpha,\xi_v\right]\widehat{\theta}(\tilde{y}_1;\alpha,\xi_w)\right]=\mathrm{E}\left[\widehat{\theta}(\tilde{y}_1;\alpha,\xi_v)\cdot\widehat{\theta}(\tilde{y}_1;\alpha,\xi_w)\right]. \quad \text{(A60)}
$$

When $\xi_w=\xi_v$ (unbiased case), it is easy to verify that

$$
\lim_{N\to\infty}\mathrm{E}\left[\sum_{n=1}^{N}(\tilde{v}-\tilde{p}_2)Z^o_{2,n}|\xi_w=\xi_v\right]=\lim_{N\to\infty}\frac{N(1-\beta_2\lambda_2)^2}{\lambda_2(N+1)^2}\mathrm{E}[\widehat{\theta}(\tilde{y}_1;\alpha,\xi_v)^2]=0. \quad \text{(A61)}
$$

Arbitrageurs eventually compete away their aggregate trading profit if $\xi_w=\xi_v$ and $N\to\infty$.

In the symmetric equilibrium of Theorem 2 where arbitrageurs use the same robust LASSO strategy $Z_{2,n}(\tilde{y}_1;\alpha,K(\xi_w))$, the expectation of their total profit is derived to be

$$
\begin{aligned}
& \mathrm{E}\left[\sum_{n=1}^{N}(\tilde{v}-\tilde{p}_2)Z_{2,n}(\tilde{y}_1;\alpha,K(\xi_w))\right] \\
= \;& \alpha(1-\beta_2\lambda_2)^2\mathrm{E}\left[\frac{N(N+1)(\tilde{v}-\lambda_1\tilde{y}_1)-\alpha N^2\widehat{\theta}^{\text{lasso}}(\tilde{y}_1;\xi_w)}{(N+1)}\cdot\frac{\widehat{\theta}^{\text{lasso}}(\tilde{y}_1;\xi_w)}{(N+1)\lambda_2}\right] \\
= \;& \alpha(1-\beta_2\lambda_2)^2\frac{\mathrm{E}\left[N(N+1)(\tilde{\theta}-\alpha\widehat{\theta}^{\text{lasso}}+\alpha\widehat{\theta}^{\text{lasso}})\widehat{\theta}^{\text{lasso}}-\alpha N^2(\widehat{\theta}^{\text{lasso}})^2\right]}{(N+1)^2\lambda_2} \\
= \;& \alpha(1-\beta_2\lambda_2)^2\frac{N\mathrm{E}\left[(\widehat{\theta}(\tilde{y}_1;\alpha,\xi_v)-\alpha\widehat{\theta}^{\text{lasso}})\widehat{\theta}^{\text{lasso}}\right]}{(N+1)\lambda_2}+\alpha^2(1-\beta_2\lambda_2)^2\frac{N\mathrm{E}\left[(\widehat{\theta}^{\text{lasso}})^2\right]}{(N+1)^2\lambda_2}, \quad \text{(A62)}
\end{aligned}
$$

where we have used Eq. (29) and the following result similar to (A60),

$$
\mathrm{E}\left[\tilde{\theta}\cdot\widehat{\theta}^{\text{lasso}}(\tilde{y}_1;\alpha,\xi_w)\right]=\mathrm{E}\left[\mathrm{E}\left[\tilde{\theta}\middle|\tilde{y}_1;\alpha,\xi_v\right]\widehat{\theta}^{\text{lasso}}\right]=\mathrm{E}\left[\widehat{\theta}(\tilde{y}_1;\alpha,\xi_v)\cdot\widehat{\theta}^{\text{lasso}}(\tilde{y}_1;\alpha,\xi_w)\right]. \quad \text{(A63)}
$$

Given the expressions of (19), (20), (30), and (31), one can further verify that

$$\mathrm{E}\left[[\widehat{\theta}(\tilde{y}_1;\alpha,\xi_v)-\alpha\widehat{\theta}^{\text{lasso}}]\widehat{\theta}^{\text{lasso}}\right]=\alpha\mathrm{E}\left[[\widehat{v}(\tilde{y}_1;\alpha=1,\xi_v)-\widehat{v}^{\text{lasso}}]\widehat{\theta}^{\text{lasso}}\right], \tag{A64}$$

where we have use the scaling property $\widehat{\theta}(\tilde{y}_1;\alpha,\xi_v)=\alpha(\widehat{v}[\tilde{y}_1;\alpha=1,\xi_v)-\lambda_1\tilde{y}_1]$ and the result that $\widehat{\theta}^{\text{lasso}}=(\widehat{v}^{\text{lasso}}-\lambda_1\tilde{y}_1)\mathbf{1}_{|\tilde{y}_1|>K}$ must satisfy $(\widehat{v}^{\text{lasso}}-\lambda_1\tilde{y}_1)\cdot\widehat{\theta}^{\text{lasso}}=(\widehat{\theta}^{\text{lasso}})^2$. In the limit $N\to\infty$, we find that the expected total profit Eq. (A62) is strictly positive:

$$\lim_{N\to\infty}\mathrm{E}\left[\sum_{n=1}^{N}(\tilde{v}-\tilde{p}_2)Z_{2,n}\right]=\frac{\alpha^2(1-\beta_2\lambda_2)^2}{\lambda_2}\mathrm{E}\left[[\widehat{v}(\tilde{y}_1;\xi_v)-\widehat{v}^{\text{lasso}}(\tilde{y}_1;\xi_w)]\cdot\widehat{\theta}^{\text{lasso}}(\tilde{y}_1;\xi_w)\right]>0. \tag{A65}$$

Here, we have used the notation $\widehat{v}(\tilde{y}_1;\xi_v)$ to stand for $\widehat{v}(\tilde{y}_1;\alpha=1,\xi_v)$. When $0<\xi_w\le\xi_v$ and $\alpha\in(0,1]$, one can verify that $\widehat{v}(\tilde{y}_1;\xi_v)>\widehat{v}^{\text{lasso}}(\tilde{y}_1;\xi_w)$ for $y_1>0$ and by symmetry, we also have $\widehat{v}(\tilde{y}_1;\xi_v)<\widehat{v}^{\text{lasso}}(\tilde{y}_1;\xi_w)$ for $y_1<0$. This means that $\widehat{v}(\tilde{y}_1;\xi_v)-\widehat{v}^{\text{lasso}}(\tilde{y}_1;\xi_w)$ and $\widehat{\theta}^{\text{lasso}}(\tilde{y}_1;\xi_w)$ have the same sign for $|\tilde{y}_1|>K(\xi_w)$, thus proving the positive sign of Eq. (A65) when $0<\xi_w\le\xi_v$. Because competition drives down profits, it also implies that Eq. (A62) is strictly positive for any $N\ge 1$ when $0<\xi_w\le\xi_v$.

We have used Monte-Carlo simulations to verify the analytical results in Theorem 4. It is also of interest to compare these two strategies in other performance measures (Table 1).

Table 1. Performance comparison based on Monte-Carlo simulations with $\alpha=1$.

$\xi_w=\xi_v, N=10$	Optimal $Z^o_{2,n}(y_1;\xi_w)$	Robust $Z_{2,n}(y_1;K(\xi_w))$
Gain-Loss Ratio	0.9545	1.3403
Profit Std Dev	0.1882	0.1107
Profit Skewness	3.9425	8.3697
Win-Loss Ratio	1.0011	1.5778
Ave Profit Per Trade	0.0014	0.1101
$\xi_w=0.9\xi_v, N=10$	Optimal $Z^o_{2,n}(y_1;\xi_w)$	Robust $Z_{2,n}(y_1;K(\xi_w))$
Gain-Loss Ratio	0.9959	1.1266
Profit Std Dev	0.1936	0.1503
Profit Skewness	5.0032	23.724
Win-Loss Ratio	1.0343	1.1593
Ave Profit Per Trade	-0.0024	0.2694

A.10 Proof of Proposition 3

When $N \to \infty$, if there is ever a finite mass, denoted $\phi \in (0,1]$, of arbitrageurs constrained by model risk as in Theorem 2, then the asset price at $t=2$ will be

$$\begin{aligned}\tilde{p}_2 &= \lambda_1\tilde{y}_1 + \lambda_2\Bigg[\beta_2(\tilde{v}-\lambda_1\tilde{y}_1) + \sum_{n=1}^{(1-\phi)N} Z^o_{2,n}(\tilde{y}_1;\alpha,\xi_w) + \sum_{n=1}^{\phi N} Z_{2,n}(\tilde{y}_1;\alpha,K(\xi_w)) + \tilde{u}_2\Bigg] \\ &= \lambda_1\tilde{y}_1 + \lambda_2\Bigg[\beta_2(\tilde{v}-\lambda_1\tilde{y}_1) + \sum_{n=1}^{N} Z^o_{2,n}(\tilde{y}_1;\alpha,\xi_w) + \sum_{n=1}^{\phi N}(Z_{2,n} - Z^o_{2,n}) + \tilde{u}_2\Bigg] \\ &\to \beta_2\lambda_2\tilde{v} + (1-\beta_2\lambda_2)\left(\widehat{v}(\tilde{y}_1;\alpha,\xi_w) + \phi\left[\alpha\widehat{\theta}^{\text{lasso}}(\tilde{y}_1;\xi_w) - \widehat{\theta}(\tilde{y}_1;\alpha,\xi_w)\right]\right) + \lambda_2\tilde{u}_2.\end{aligned}$$

We will keep using $\widehat{v}(\tilde{y}_1;\xi)$ to stand for $\widehat{v}(\tilde{y}_1;\alpha=1,\xi)$ and use $\widehat{\theta}(\tilde{y}_1;\xi)$ for $\widehat{\theta}(\tilde{y}_1;\alpha=1,\xi)$. The expectation of $\tilde{p}_2 - \tilde{v}$ conditional on the price history $\{\tilde{p}_1 = \lambda_1\tilde{y}_1, p_0\}$ is

$$\begin{aligned}&\lim_{N\to\infty} \mathrm{E}[\tilde{p}_2 - \tilde{v}|\tilde{p}_1, p_0] = \lim_{N\to\infty} \mathrm{E}[\lambda_1\tilde{y}_1 + \lambda_2\tilde{y}_2 - \tilde{v}|\tilde{y}_1] \\ &= (1-\beta_2\lambda_2)\left(\widehat{v}(\tilde{y}_1;\alpha,\xi_w) - \mathrm{E}[\tilde{v}|\tilde{y}_1;\alpha,\xi_v]\right) + \phi(1-\beta_2\lambda_2)\left[\alpha\widehat{\theta}^{\text{lasso}}(\tilde{y}_1;\xi_w) - \widehat{\theta}(\tilde{y}_1;\alpha,\xi_w)\right] \\ &= (1-\beta_2\lambda_2)\left(\widehat{v}(\tilde{y}_1;\alpha,\xi_w) - \widehat{v}(\tilde{y}_1;\alpha,\xi_v)\right) + \phi(1-\beta_2\lambda_2)\left[\alpha\widehat{\theta}^{\text{lasso}}(\tilde{y}_1;\xi_w) - \widehat{\theta}(\tilde{y}_1;\alpha,\xi_w)\right] \\ &= (1-\beta_2\lambda_2)\left(\left[\widehat{v}(\tilde{y}_1;\alpha,\xi_w) - \widehat{v}(\tilde{y}_1;\alpha,\xi_v)\right] + \alpha\phi\left[\widehat{\theta}^{\text{lasso}}(\tilde{y}_1;\xi_w) - \widehat{\theta}(\tilde{y}_1;\xi_w)\right]\right) \\ &= \alpha(1-\beta_2\lambda_2)\left(\left[\widehat{v}(\tilde{y}_1;\xi_w) - \widehat{v}(\tilde{y}_1;\xi_v)\right] + \phi\left[\widehat{\theta}^{\text{lasso}}(\tilde{y}_1;\xi_w) - \widehat{\theta}(\tilde{y}_1;\xi_w)\right]\right). \qquad \text{(A66)}\end{aligned}$$

In deriving Eq. (A66), we have used the scaling property that $\widehat{\theta}(\tilde{y}_1;\alpha,\xi_w) = \alpha\widehat{\theta}(\tilde{y}_1;\alpha=1,\xi_w)$ and the observation that $\widehat{v}(\tilde{y}_1;\alpha,\xi_w) - \widehat{v}(\tilde{y}_1;\alpha,\xi_v) = \alpha\left[\widehat{v}(\tilde{y}_1;\alpha=1,\xi_w) - \widehat{v}(\tilde{y}_1;\alpha=1,\xi_v)\right]$, which follows from the expression of (20). Eq. (A66) is not equal to zero almost everywhere (for almost any realizations of $\tilde{y}_1$), unless both $\xi_w = \xi_v$ and $\phi = 0$ hold simultaneously.

Note that the economy in either Theorem 1 or Theorem 2 can hold an infinite number of arbitrageurs when the following inequality holds: $\alpha + (1-\alpha)\beta_1\lambda_1 < \frac{2\beta_1\sqrt{\lambda_1\lambda_2}}{1-\beta_2\lambda_2}$. This is from the equilibrium condition (23) in Proposition 1 by taking the limit $N \to \infty$.

Figure 16 plots the aggregate trading profiles of arbitrageurs when they follow the robust LASSO strategy. The red solid line is for the homogeneous case when they use identical trading thresholds $K(\xi_v)$. The black dash-dot line is for the heterogeneous case when they use different trading thresholds $K(\xi_{w,n})$ for $n = 1, ..., N$. It has a narrower no-trade region determined by the most optimistic trader whose effective prior is $\xi^*_w := \max\{\xi_{w,1}, ..., \xi_{w,N}\}$. We impose $\frac{1}{N}\sum_{n=1}^{N}\xi^{-1}_{w,n} = \xi^{-1}_v$ so that the two aggregate trading strategies converge.

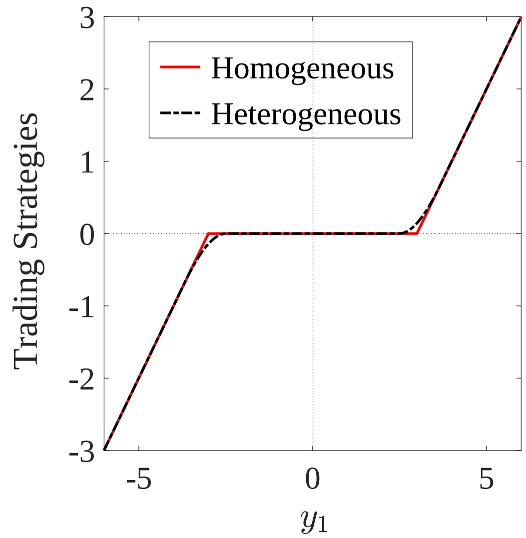

Figure 16. Comparison of the robust strategy $Z_{2,n}(y_1; \alpha, K(\xi_v))$ homogeneous for each n and the population average of the robust strategy over traders with heterogeneous thresholds, $\frac{1}{N}\sum_{n=1}^{N} Z_{2,n}(y_1; \alpha, K(\xi_{w,n}))$, where $\frac{1}{N}\sum_{n=1}^{N} \xi_{w,n}^{-1} = \xi_v^{-1}$ is imposed for a fair comparison.

A.11 Proof Proposition 5

Consider $J \geq 1$ stocks whose liquidation values have independent Laplacian-Gaussian mixture distributions, $\tilde{v}_j \sim \mathcal{LG}(\alpha_j, \xi_j)$, for $j = 1, ..., J$. Suppose arbitrageurs solve the same problem as in Theorem 2 for each asset based on their uncertain fat-tail prior $\tilde{v}_j \sim \mathcal{LG}(\alpha_j, \tilde{\xi}_j)$. When (C1)-(C3) hold for each asset, their robust strategy at $t = 2$ is similar to Eq. (28):

$$Z_{2,n}(y_{1,j}; \alpha_j, \xi_{w,j}) = \frac{\alpha_j(1-\beta_{1,j}\lambda_{1,j})(1-\beta_{2,j}\lambda_{2,j})}{\beta_{1,j}\lambda_{2,j}(N+1)} \left[y_{1,j} - \text{sign}(y_{1,j})K_j(\xi_{w,j})\right] \mathbf{1}_{|y_{1,j}|>K_j(\xi_{w,j})} \quad \text{(A67)}$$

which is a soft-thresholding function of $y_{1,j}$. With $p_{1,j} = 0$ and $p_{1,j} = \lambda_{1,j} y_{1,j}$, we can define the LASSO estimates $\widehat{\theta}_j^{\text{lasso}}(p_{1,j}; \xi_{w,j})$ of the pricing error for each asset $j \in \{1, ..., J\}$ as

$$\widehat{\theta}_j^{\text{lasso}} := \arg\min_{\theta_j} \frac{1}{2}\left|p_{1,j} - \frac{\beta_{1,j}\lambda_{1,j}\theta_j}{1-\beta_{1,j}\lambda_{1,j}}\right|^2 + \left(\frac{\lambda_{1,j}\sigma_{u,j}}{1-\beta_{1,j}\lambda_{1,j}}\right)^2 \frac{|\theta_j|}{\xi_{w,j}} = \frac{1-\beta_{1,j}\lambda_{1,j}}{\beta_{1,j}\lambda_{1,j}} \mathcal{S}(p_{1,j}; \lambda_{1,j}K_j). \quad \text{(A68)}$$

By Theorem 3, the robust strategy for each asset $j \in \{1, ..., J\}$ is a LASSO strategy:

$$Z_{2,n}(p_{1,j}; \alpha_j, \xi_{w,j}) = \frac{\alpha_j(1-\beta_{2,j}\lambda_{2,j})}{\lambda_{2,j}(N+1)} \widehat{\theta}_j^{\text{lasso}}(p_{1,j}; \xi_{w,j}). \quad \text{(A69)}$$

By proposition 4, if we consider the uncertainty of $\tilde{\alpha}_j$, we can simply replace α_j by its prior mean $\overline{\alpha}_j$ in the above equation and the new LASSO strategy will be $Z_{2,n}(p_{1,j}; \overline{\alpha}_j, \xi_{w,j})$.

A.12 Proof of Proposition 6

All other things being equal, we replace Eq. (10) by the mixture Gaussian prior below:

$$f(v;\alpha,\zeta) = \frac{\alpha}{\sqrt{2\pi\zeta^2}}\exp\left(-\frac{v^2}{2\zeta^2}\right) + \frac{1-\alpha}{\sqrt{2\pi\sigma_v^2}}\exp\left(-\frac{v^2}{2\sigma_v^2}\right). \tag{A70}$$

Arbitrageurs know α but are uncertain about the volatility ζ of the Gaussian shocks. Since Bayesian learning will maintain the simple scaling property about α, it suffices to consider the case that $\alpha = 1$. Each arbitrageur's prior in this case becomes $\tilde{v} \sim \mathcal{N}(0, \tilde{\zeta}^2)$, where $\tilde{\zeta} \in [\zeta_L, \zeta_H]$. For any fixed Gaussian prior $\tilde{\zeta} = \zeta$, arbitrageurs believe that $\tilde{y}_1 = \beta_1\tilde{v} + \tilde{u}_1 \sim \mathcal{N}\left(0, (\beta_1\zeta)^2 + \sigma_u^2\right)$. Their posterior belief about $\tilde{v}$ conditional on $\tilde{y}_1$ is

$$f(v|y_1) = \frac{f(y_1|v)f(v;\zeta)}{f(y_1)} = \frac{1}{2\pi\zeta\sigma_u f(y_1)}\exp\left[-\frac{(y_1-\beta_1 v)^2}{2\sigma_u^2} - \frac{v^2}{2\zeta^2}\right]. \tag{A71}$$

By projection theorem, they will use the linear estimator under the Gaussian prior,

$$\widehat{v}^{\text{ridge}}(y_1;\zeta) = \mathrm{E}[\tilde{v}|y_1,\zeta] = \frac{\beta_1\zeta^2 y_1}{(\beta_1\zeta)^2 + \sigma_u^2} = \frac{y_1}{\beta_1 + \sigma_u^2/(\beta_1\zeta^2)}, \tag{A72}$$

which is the simplest version of *ridge regression* (Hastie et al. (2009)) with an l_2 norm penalty. For any given ζ, the optimal strategy of arbitrageurs is always a linear function of y_1:

$$Z_{2,n}^o(y_1;\zeta) = \frac{1-\beta_2\lambda_2}{\lambda_2(N+1)}(\widehat{v}^{\text{ridge}} - \lambda_1 y_1) = \frac{\beta_1(1-\beta_1\lambda_1)\zeta^2 - \lambda_1\sigma_u^2}{(\beta_1\zeta)^2 + \sigma_u^2}\cdot\frac{1-\beta_2\lambda_2}{\lambda_2(N+1)}y_1. \tag{A73}$$

Any uncertainty about the prior ζ only changes the slope of this linear strategy. Therefore, the robust strategy should be linear under the max-min choice criteria. It is easy to verify that $Z_{2,n}^o(y_1;\zeta=\sigma_v) = 0$ since the price is supposed to be efficient when $\zeta = \sigma_v$. Hence, if $\zeta_L \le \sigma_v \le \zeta_H$, then $Z_{2,n}^o(y_1;\zeta_H)$ is upward sloping and $Z_{2,n}^o(y_1;\zeta_L)$ is downward sloping such that the max-min strategy is no trade at all. If $\sigma_v < \zeta_L \le \zeta_H$, then the max-min strategy is the upward sloping linear strategy $Z_{2,n}^o(y_1;\zeta_L)$. If $\zeta_L \le \zeta_H < \sigma_v$, then the max-min strategy is the downward-sloping linear strategy $Z_{2,n}^o(y_1;\zeta_H)$. In sum, the max-min robust strategy is

$$Z_{2,n}(y_1;\zeta) = \begin{cases} Z_{2,n}^o(y_1;\zeta_L), & \text{if } \sigma_v < \zeta_L \le \zeta_H, \\ 0, & \text{if } \zeta_L \le \sigma_v \le \zeta_H, \\ Z_{2,n}^o(y_1;\zeta_H), & \text{if } \zeta_L \le \zeta_H < \sigma_v. \end{cases} \tag{A74}$$

The above strategy is always a linear function of y_1, without any trading threshold.

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