ECO 751 Public Economics
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Professor Hoyt

Lecture 11: Optimal Taxation

1. The Motivation

The motivation of optimal taxation, specifically the theory of optimal commodity taxtion, is based on the question of what is the appropriate policy for setting commodity tax rates that A.C. Pigou proposed to Frank Ramsey. The object is to raise some given amount of revenue in a way that minimizes the cost of collecting the revenue. More precisely the set of taxes is chosen to maximize the utility of the residents, or, equivalently, minimize excess burden.

2. Optimal Taxation with Exogenous Income

Intuitively it might seem that the most desirable tax policy would be an equal ad valorem tax rate on all goods. Letting p_i be the constant producer price of good *i* and τ, the ad-valorem tax rate we have

$$\frac{\frac{\partial U^t}{\partial x_i}}{\frac{\partial U^t}{\partial x_j}} = MRS_{ij}^t = \frac{(1+\tau)p_i}{(1+\tau)p_j} = \frac{p_i}{p_j} \tag{1}$$

where U^t is the utility of individual t and x, is the consumption of good i. Then the marginal rate of substitution between any two goods *i* and τ is equal to the ratio of the costs of production.

Alternatively we know that since the tax is assessed at a rate of τ on all commodities by the budget constraint we have

$$y_i = \sum_{i=1}^{n}(1+\tau)p_i x_i = \sum_{i=1}^{n} p_i x_i + \tau\sum_{i=1}^{n} p_i x_i \tag{2}$$

where y_i is the income of individual i. Then we can rewrite (2) as

$$y - R = y - \left(\frac{\tau}{(1+\tau)}\right)y = \sum_{i=1}^{n} p_i x_i \tag{2'}$$

where R is tax revenue. The constant ad-valorem tax rate is equivalent to a proportional income tax (on exogenous income) with a rate of τ/(1+τ).

More formally, we can model this problem as choosing commodity tax rates to maximize the utility of a representative individual while being constrained to raise revenue of *R*. Letting τ_i be the per unit tax on commodity i;

- $q_i = p_i + \tau_i$ is the price the consumer pays for commodity i;
- $x_i(\underline{q},y)$ is the demand for commodity *i* where y is income;
- $V(\underline{q},y)$ is the indirect utility function; and
- R is the revenue requirement.

Then the optimal tax solves:

$$\underset{\underline{\tau}}{Maximize}\ V(\underline{q}, y)$$
$$s.t. \sum_{i=1}^{n} \tau_i x_i(\underline{q}, y) = R \tag{3}$$

which gives the La Grangian of

$$L = V(\underline{q}, y) - \mu\left(R - \sum_{i=1}^{n} \tau_i x_i(\underline{q}, y)\right) \tag{4}$$

The first order conditions are:

$$\frac{\partial L}{\partial \tau_i} = \frac{\partial V}{\partial q_i} + \mu\left(\sum_{j=1}^{n} \tau_j \frac{\partial x_j}{\partial q_i} + x_i\right) = 0,\ i = 1,\ldots,n \tag{5}$$

Using Roy's identity $\left(x_i = -\dfrac{\frac{\partial V}{\partial q_i}}{\frac{\partial V}{\partial y}} \right)$ this can be rewritten as

$$\frac{\partial L}{\partial \tau_i} = -\lambda x_i + \mu x_i + \mu\left(\sum_{j=1}^{n} \tau_j \frac{\partial x_j}{\partial q_i}\right) = 0,\ i = 1,\ldots,n \tag{5'}$$

where λ equals ∂V/∂y*,* the marginal utility of income. Recall from the Slutaky decomposition that:

$$\frac{\partial x_j}{\partial q_i} = \frac{\partial h_j}{\partial q_i} - x_i \frac{\partial x_j}{\partial q_i} \tag{6}$$

where $h_j(\underline{q}, U)$ is the Hicksian (compensated) demand for good *j*. Then using the Slutsky decomposition in *(5')* gives

$$\frac{\partial L}{\partial \tau_i} = -\lambda x_i + \mu x_i + \mu\left(\sum_{j=1}^{n} \tau_i \left(\frac{\partial h_j}{\partial q_i} - x_i \frac{\partial x_j}{\partial q_i}\right)\right) = 0,\ i = 1,\ldots,n \tag{5''}$$

Let $\alpha = \lambda + \mu \sum_{j=1}^{n} \tau_j \dfrac{\partial x_j}{\partial y}$ (7)

where α is the marginal social utility of income. This is the direct effect of a tax increase on utility, X and the indirect effect of changes in revenue. Then using (7) in *(5")* and rewriting gives

$$(\mu - \alpha)x_i = \mu\left(-\sum_{j=1}^{n} \tau_j \frac{\partial h_j}{\partial q_i}\right) = 0,\ i = 1,\ldots,n \tag{8}$$

and finally we have

$$-\sum_{j=1}^{n}\tau_j \frac{\partial h_j}{\partial q_i} = \left[\frac{(\mu-\alpha)}{\mu}\right]x_i,\ i=1,\ldots,n \tag{8'}$$

The term μ - α is the marginal excess burden of the tax.

The Solution

Let the proposed solution be $\tau_j = \theta p_j$, a proportional ad-valorem tax. Then (8') becomes

$$-\theta\sum_{j=1}^{n} p_j \frac{\partial h_j}{\partial q_i} = \left[\frac{(\mu-\alpha)}{\mu}\right]x_i,\ i=1,\ldots,n \tag{9}$$

But $h_i(\theta \underline{p},U) = h_i(p,U)$ by the homogeneity of Hicksian demands. This means that $\frac{\partial h_j(\theta \underline{p},U)}{\partial q_i} = \frac{\partial h_j(\underline{p},U)}{\partial q_i}$.

Then we can show the left side of (9) equals zero in the following way:

$$\sum_{j=1}^{n} p_j h_j(\underline{p},U) = e(\underline{p},U) \tag{10}$$

Equation (10) is simply the definition of the expenditure function. Then differentiating (10) with respect to q_i gives

$$\sum_{j=1}^{n} p_j \frac{\partial h_j}{\partial q_i} + h_i = \frac{\partial e}{\partial q_i} \tag{11}$$

but $h_i = \frac{\partial e}{\partial q_i}$ by Shephard's Lemma so the term $\sum_{j=1}^{n} p_j \frac{\partial h_j}{\partial q_i} = 0$ as well. Then the right side of (9) also equals zero — this means there is no excess burden. This should be no surprise as we demonstrated earlier that the same ad valorem tax on all commodities was equivalent to a lump sum tax when all commodities are taxed.

Result 1: If all commodities can be taxed then proportional taxation is optimal.

Thus the only optimal commodity tax problem that is interesting must involve some commodity that is not taxed. What might this good be? Leisure as it cannot be directly identified and therefore cannot be directly taxed. If taxed are only placed on goods purchased in the market, then clearly leisure escapes taxation. Nor can it be taxed through taxes or subsidies on labor as a subsidy of *s* on labor is equivalent to lowering the ad valorem rate on all goods by *s*.

3. A Simple Model of Optimal Taxation

The preceding section demonstrated that any tax structure other than having the same ad valorem rates on all commodities can only be optimal if some commodity is not taxed. As we can not identify labor effort and, therefore leisure, leisure can be the omitted commodity from the tax base. In this section we present a simple single-consumer model to generate a simple rule for optimal taxation when all goods are neither gross substitutes or complements with any other goods. We then relax this restrictive assumption in the next section to generate the more traditional optimal tax rules.

A. Measures of Excess Burden

Before we attempt to formally derive any optimal tax rules recall from last lecture that the excess burden of a single tax on good *i* is approximated by:

$$EB(\tau_i) = \frac{1}{2}\varepsilon_i \tau_i \frac{x_i}{q_i} \tag{12}$$

where ε_i is the own-price elasticity of good *i.* Then the amount of excess burden from a single tax depends on:

1) the magnitude of the tax; and
2) the elasticity of demand.

Figure la shows that the more elastic the demand for a good, the greater the excess burden for any given tax rate and quantity. Figure lb shows that doubling the tax on a good will increase the excess burden by a factor of four.

Figure 1a

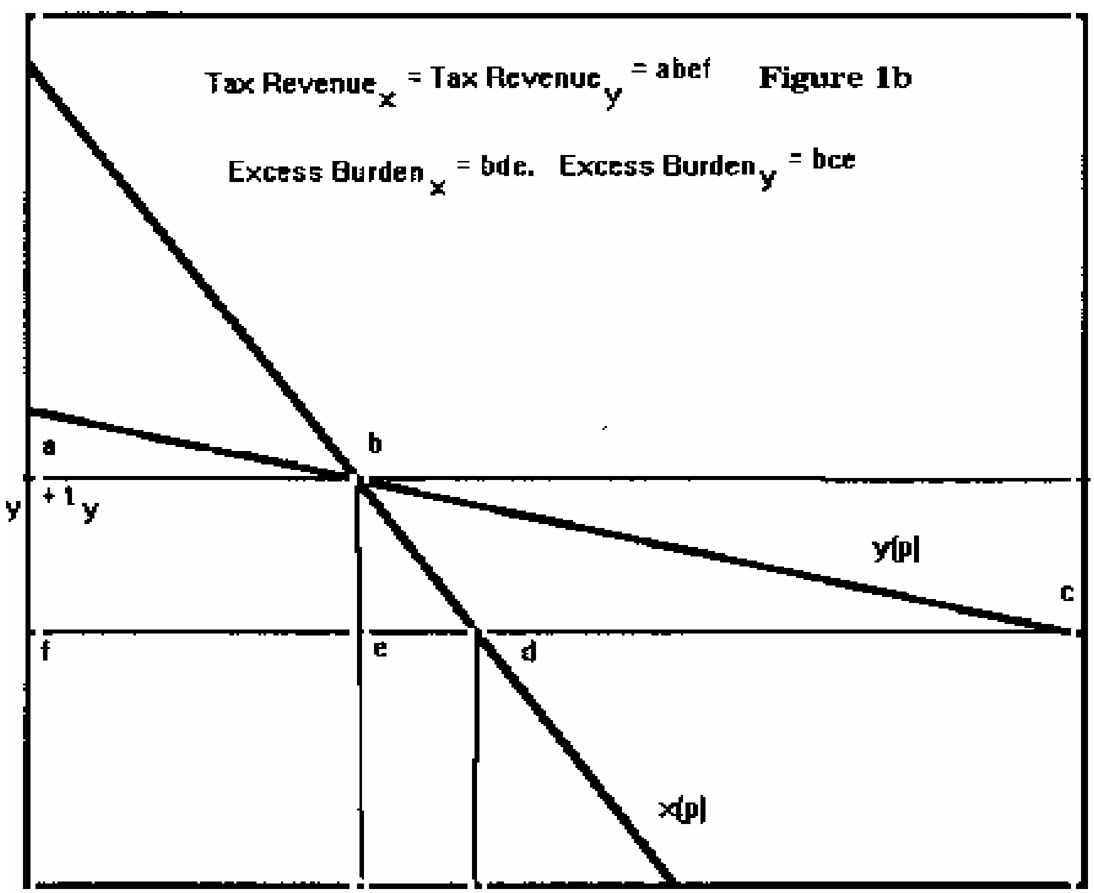
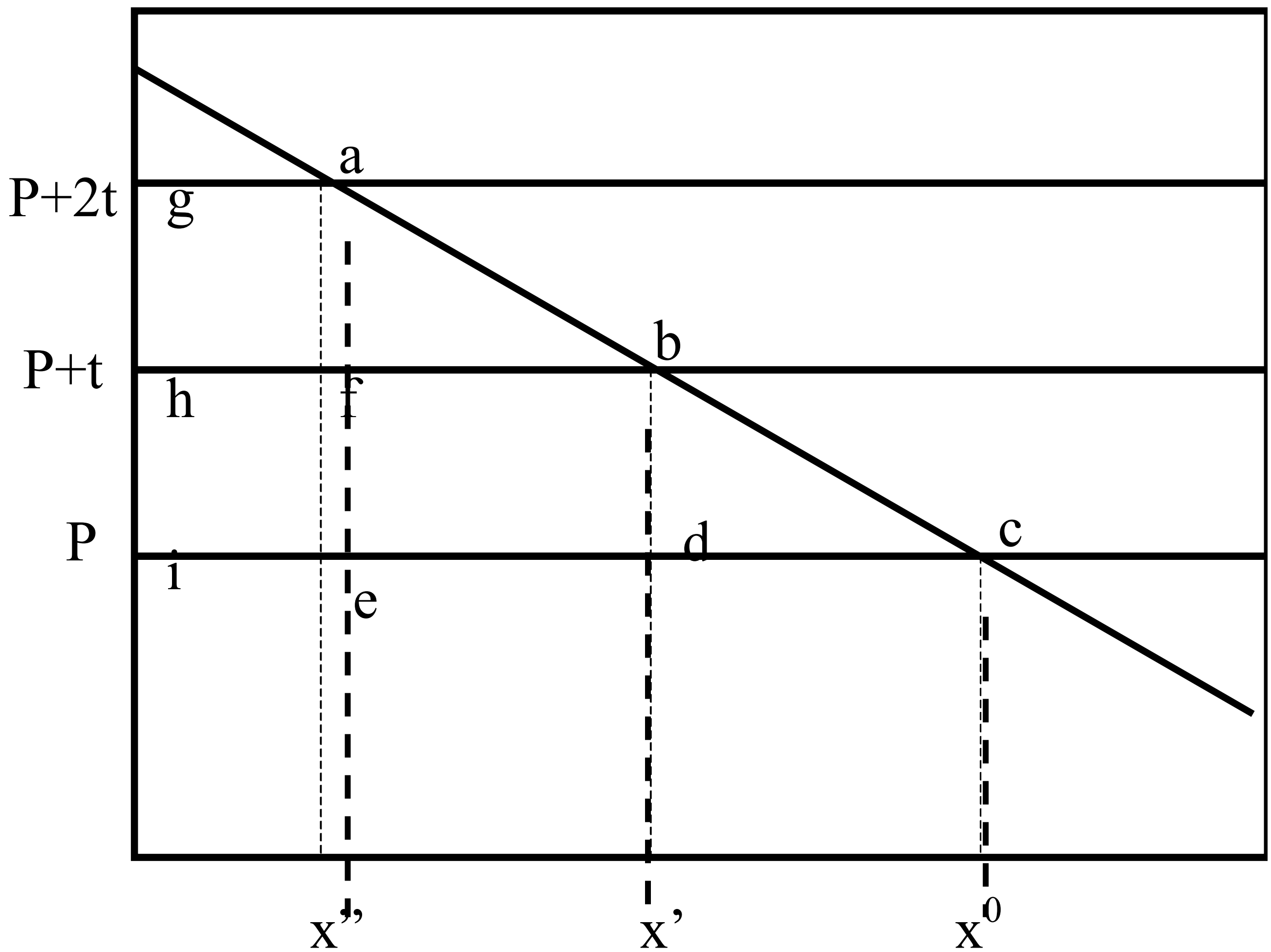


Figure 1b



In *Figure 1b* we have excess burden with a tax of t equal to acd but with a tax of 2t it becomes ace.

B. The Objective

What might be the rule for choosing the tax rates? In addition to excess burden of a tax, we must also consider the revenue collected from a tax.

$$R_i = \tau_i x_i(q_i) \text{ where } x_i(q_i) = x_i^o + \varepsilon_i \frac{\tau_i}{q_i} x_i^0 \tag{13}$$

We might consider relative excess burden:

$$\frac{EB(\tau_i)}{R_i} = \frac{-\frac{1}{2}\varepsilon_i \tau_i^2 \frac{x_i^0}{p_i}}{\tau_i\left(x_i^o + \varepsilon_i \tau_i \frac{x_i^0}{p_i}\right)} = \frac{-\frac{1}{2}}{\left[\frac{p_i}{\varepsilon_i \tau_i} + 1\right]} \tag{14}$$

Then we could have as a rule that relative excess burden is equal among goods or

$$\frac{EB(\tau_i)}{R_i} = \frac{-\frac{1}{2}}{\left[\frac{p_i}{\varepsilon_i \tau_i} + 1\right]} = \frac{-\frac{1}{2}}{\left[\frac{p_j}{\varepsilon_j \tau_j} + 1\right]} = \frac{EB(\tau_j)}{R_j} \tag{15}$$

or

$$\frac{\tau_i}{p_i} \bigg/ \frac{\tau_j}{p_j} = \frac{\varepsilon_j}{\varepsilon_i} \tag{16}$$

This is an "inverse" elasticity rule that says that the tax rate as fraction of net price is inversely related to the tax rates.

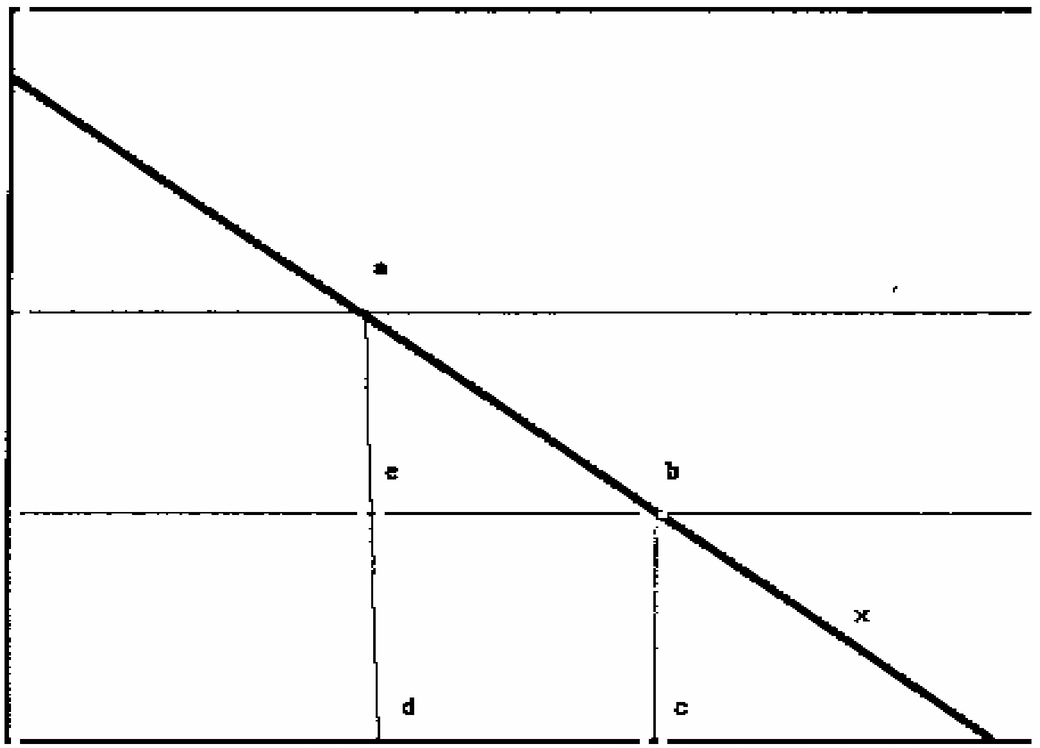
Alternatively, let us consider minimizing excess burden across the economy given our revenue requirement. Then the objective is

$$\underset{\tau_1,\ldots,\tau_n}{Minimize} \sum_{i=1}^{n} \left[\int_{x_i(p_i+\tau_i)}^{x_i(p_i)} q_i(x_i)dx_i - p_i\left[x_i(p_i) - x_i(p_i+\tau_i)\right]\right]$$
$$s.t. \ \sum_{i=1}^{n} \tau_i x(q_i) \geq R \tag{17}$$

where $q_i(x_i)$ is the marginal benefit for the x_i^{th} unit. We assume that the demand for good x_i depends on only the gross price of x_i q_i and $\frac{\partial x_j}{\partial q_i} = 0, \ j \neq i$.

Graphically the excess burden as we formulated it in (17) is shown in Figure 2. The excess burden is the loss in net benefit, abcd, (the integral in (17)) less what the consumer was paying, ebcd, making excess burden the triangle, abe.

Figure 2



Then the LaGrangian associated with (17) is simply

$$L = \sum_{i=1}^{n}\left[\int_{x_i(p_i+\tau_i)}^{x_i(p_i)} q_i(x_i)dx_i - p_i\left[x_i(p_i) - x_i(p_i+\tau_i)\right]\right] + .\ \lambda\left(\sum_{i=1}^{n}\tau_i x(q_i) - R\right) \qquad (18)$$

The first order condition with respect to τ_i is

$$\frac{\partial L}{\partial \tau_i} = -q_i\frac{\partial x_i}{\partial q_i}(q_i) + p_i\frac{\partial x_i}{\partial q_i}(q_i) + \lambda\left[x_i(q_i) + \tau_i\frac{\partial x_i}{\partial q_i}(q_i)\right] = 0 \qquad (19)$$

which simplifies to:

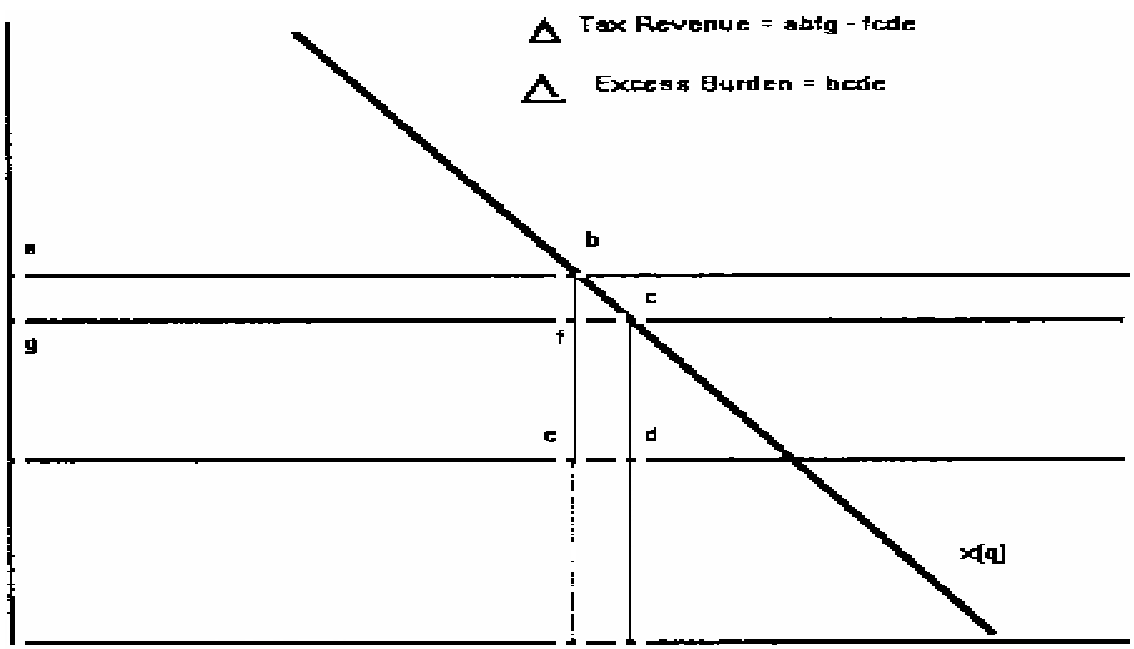
$$\frac{\partial L}{\partial \tau_i} = -\tau_i\frac{\partial x_i}{\partial q_i}(q_i) + \lambda\left[x_i(q_i) + \tau_i\frac{\partial x_i}{\partial q_i}(q_i)\right] = 0 \qquad (19')$$

The first order condition can be rearranged to obtain:

$$\tau_i\frac{x_i}{\frac{\partial x_i}{\partial q_i}} = \frac{\tau_i}{(\tau_i + p_i)}\frac{1}{\varepsilon_i} = -\frac{1-\lambda}{\lambda} \qquad (20)$$

This is the inverse elasticity rule that says that the tax rate should be inversely related to the price elasticity of demand. Intuitively our result says that the marginal excess burden ($\partial EB/\partial \tau_i = -\tau_i(\partial x_i/\partial q_i)$) divided by the marginal revenue ($\partial R/\partial \tau_i = x_i + \tau_i(\partial x_i/\partial q_i)$) should be equal in all markets (equal to λ) where marginal excess burden and marginal tax revenue are illustrated in Figure 3.

Figure 3



4. *Optimal Taxation in a Single Consumer World*

We now wish to consider a more general specification of the optimal tax problem. Specifically, we relax the assumption that the cross-price elasticities for all goods are equal to zero. We shall also formulate the problem in the more traditional second-best approach of maximizing the utility of a representative individual (our single consumer) by choosing our instruments, in this case tax rates, subject to a revenue constraint. As tax rates affect the prices consumers face, it is natural to use the indirect utility function. We continue with the assumption that the marginal cost of good *i* is p, regardless of the amount of production. Then the producer's profits are unaffected by the tax. Then our problem is the same as we presented in *Section 2* with one important difference -- in this case we can not tax all goods but only a subset of them. Assume there are k goods in all but only n, k > n, can be taxed. Then given our revenue requirement of R we have the problem:

$$\underset{\underline{\tau}}{Maximize}\ V(\underline{q}, y)$$
$$s.t.\ \sum_{i=1}^{n} \tau_i x_i(\underline{q}, y) = R \tag{21}$$

which gives the La Grangian of

$$L = V(\underline{q}, y) - \mu\left(R - \sum_{i=1}^{n} \tau_i x_i(\underline{q}, y)\right) \tag{22}$$

The first order conditions are:

$$\frac{\partial L}{\partial \tau_i} = \frac{\partial V}{\partial q_i} + \mu\left(\sum_{j=1}^{n} \tau_j \frac{\partial x_j}{\partial q_i} + x_i\right) = 0,\ i = 1,\ldots,n \tag{23}$$

Using Roy's identity $\left(x_i = -\frac{\frac{\partial V}{\partial q_i}}{\frac{\partial V}{\partial y}} \right)$ this can be rewritten as

$$\frac{\partial L}{\partial \tau_i} = -\lambda x_i + \mu x_i + \mu\left(\sum_{j=1}^{n} \tau_j \frac{\partial x_j}{\partial q_i}\right) = 0,\ i = 1,\ldots,n \tag{23'}$$

where λ equals $\partial V/\partial y$, the marginal utility of income. Recall from the Slutsky decomposition that:

$$\frac{\partial x_j}{\partial q_i} = \frac{\partial h_j}{\partial q_i} - x_i \frac{\partial x_j}{\partial q_i} \tag{24}$$

where $h_j(\underline{q},U)$ is the Hicksian (compensated) demand for good j. Then using the Slutsky decomposition in *(5')* gives

$$\frac{\partial L}{\partial \tau_i} = -\lambda x_i + \mu x_i + \mu\left(\sum_{j=1}^{n} \tau_i \left(\frac{\partial h_j}{\partial q_i} - x_i \frac{\partial x_j}{\partial q_i}\right)\right) = 0,\ i = 1,\ldots,n \tag{23''}$$

Let $\alpha = \lambda + \mu \sum_{j=1}^{n} \tau_j \frac{\partial x_j}{\partial y}$ (25)

where α is the marginal social utility of income. This is the direct effect of a tax increase on utility, X and the indirect effect of changes in revenue. Then using (7) in *(5")* and rewriting gives

$$(\mu - \alpha)x_i = \mu\left(-\sum_{j=1}^{n} \tau_j \frac{\partial h_j}{\partial q_i}\right) = 0,\ i = 1,\ldots,n \tag{26}$$

and finally we have

$$-\sum_{j=1}^{n} \tau_j \frac{\partial h_j}{\partial q_i} = \left[\frac{(\mu - \alpha)}{\mu}\right] x_i,\ i = 1,\ldots,n \tag{27}$$

The term $\mu - \alpha$ is the marginal excess burden of the tax.

The Solution

In *Section 2* we had the proposed solution be $\tau_j = \theta p_j$, a proportional ad-valorem tax. Then (27) becomes

$$-\theta\sum_{j=1}^{n} p_j \frac{\partial h_j}{\partial q_i} = \left[\frac{(\mu-\alpha)}{\mu}\right] x_i,\ i = 1,\ldots,n \quad (28)$$

In section 2 because we taxed all the goods the left side equaled zero. But if we only taxed some of them it is not the case that the left side equals zero. Thus the proportionate tax will not eliminate excess burden and the proportionate tax will not minimize excess burden. To find an interpretable condition we use the fact that $\frac{\partial h_j}{\partial p_i} = \frac{\partial h_i}{\partial p_j}$ by the symmetry of the Slutsky matrix to express (27) as

$$\sum_{j=1}^{n} \tau_j \frac{\partial h_i}{\partial q_j} = \theta x_i,\ i = 1,\ldots,n \quad (29)$$

where $\theta = -\frac{(\mu-\alpha)}{\mu}$. Finally we can reexpress this as

$$\sum_{j=1}^{n} \frac{\tau_j}{q_j} \eta_{ij} = \theta,\ i = 1,\ldots,n \quad (29')$$

where $\eta_{ij} = \frac{\partial h_i}{\partial q_j}\frac{q_j}{h_i}$. We can interpret (27) as rule that states that taxes should be set that the marginal excess burden across all markets from a tax on good i is proportionate to the change in income (x_i) needed to compensate the individual. We can interpret (29') as an augmented elasticity rule -- in addition to using the price elasticity of demand for good i with respect to the price of i, you must consider the impact of all the other taxes on good i. Then (27) or (29') represent n equations to find the n tax rates.

6. Optimal Taxation and Labor Supply

One good we consume avoids taxation--leisure. Because we can not distinguish between whether someone has high income because he has a large endowment of labor or because he consumes little leisure. Identifying income is not equivalent to identifying leisure. An income tax is equivalent to a uniform ad valorem tax on all market goods with leisure avoiding taxation (Alternatively, an income tax could be interpreted as a lump sum tax on the endowment of labor with a subsidy to the consumption of leisure).

Given that leisure escapes taxation, how should we design our tax system to reflect this? Can we indirectly tax leisure?

A simple example: Let there be 2 market goods, 1 and 2, and leisure (minus leisure, good 0. From (29) we have

$$\tau_1 \frac{\partial h_1}{\partial q_1} + \tau_2 \frac{\partial h_1}{\partial q_2} = \theta x_1 \quad (30a)$$

and

$$\tau_1 \frac{\partial h_2}{\partial q_1} + \tau_2 \frac{\partial h_2}{\partial q_2} = \theta x_2 \tag{30b}$$

Then if $p_1 = p_2 = 1$, $q_1 = 1+\tau_1$ and $q_2 = 1+\tau_2$ we can rewriting (30) in terms of elasticities to give

$$\frac{\tau_1}{(1+\tau_1)}\eta_{11} + \frac{\tau_2}{(1+\tau_2)}\eta_{12} = \theta = \frac{\tau_1}{(1+\tau_1)}\eta_{21} + \frac{\tau_2}{(1+\tau_2)}\eta_{22} \tag{31}$$

Then we can re-express (31) as

$$\frac{\tau_1}{(1+\tau_1)} = \left[\frac{(\eta_{12} - \eta_{22})}{(\eta_{21} - \eta_{11})}\right]\frac{\tau_2}{(1+\tau_2)} \tag{31'}$$

By the definition of the expenditure function we have

$$\sum_{j=1}^{n} p_j h_j = e(\underline{p}, U) \tag{32}$$

Then differentiating with respect to τ_i gives

$$\sum_{j=1}^{n} p_j \frac{\partial h_j}{\partial q_i} + h_i = \frac{\partial e}{\partial q_i} = h_i \tag{33}$$

and simplifying gives

$$\sum_{j=1}^{n} p_j \frac{\partial h_j}{\partial q_i} = 0 \tag{34}$$

and by the symmetry of the Slutsky matrix we have $\frac{\partial h_j}{\partial p_i} = \frac{\partial h_i}{\partial p_j}$ giving

$$\sum_{j=1}^{n} p_j \frac{\partial h_i}{\partial q_j} = 0 \rightarrow \sum_{j=1}^{n} p_j \frac{1}{h_i}\frac{\partial h_i}{\partial q_j} = 0 \rightarrow \sum_{j=1}^{n} \eta_{ij} = 0. \tag{35}$$

Then for the three goods we have $\eta_{11} + \eta_{12} + \eta_{10} = 0$ and $\eta_{21} + \eta_{22} + \eta_{20} = 0$. Then we can substitute $-(\eta_{11}+ \eta_{10})$ for η_{12} and $-(\eta_{22}+ \eta_{20})$ for η_{21} in (31') to obtain

$$\frac{\tau_1}{(1+\tau_1)} = \left[\frac{-(\eta_{11} + \eta_{22}) - \eta_{10}}{-(\eta_{11} + \eta_{22}) - \eta_{20}}\right]\frac{\tau_2}{(1+\tau_2)} \tag{36}$$

If $\eta_{10} > 0$ then leisure and good 1 are substitutes. If $\eta_{10} > \eta_{20}$ then $\tau_1 < \tau_2$. The optimal policy is the one in which the complements of the untaxed good, leisure, are taxed at a higher rate. *Table 1* below gives the relative tax rate ad-valorem rate $\frac{\tau_1}{(1+\tau_1)} \Big/ \frac{\tau_2}{(1+\tau_2)}$ for $\eta_{11} = \eta_{22} = -1$ and different elasticities for η_{10} and η_{20} ranging between -1 and 1.

Table 1

Cross Price Elasticity of Good 1 and Leisure (η_{10}) \ Cross Price Elasticity of Good 2 and Leisure (η_{20})	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
-1	1.00	1.07	1.15	1.25	1.36	1.50	1.67	1.88	2.14	2.50	3.00
-0.8	0.93	1.00	1.08	1.17	1.27	1.40	1.56	1.75	2.00	2.33	2.80
-0.6	0.87	0.93	1.00	1.08	1.18	1.30	1.44	1.63	1.86	2.17	2.60
-0.4	0.80	0.86	0.92	1.00	1.09	1.20	1.33	1.50	1.71	2.00	2.40
-0.2	0.73	0.79	0.85	0.92	1.00	1.10	1.22	1.38	1.57	1.83	2.20
0	0.67	0.71	0.77	0.83	0.91	1.00	1.11	1.25	1.43	1.67	2.00
0.2	0.60	0.64	0.69	0.75	0.82	0.90	1.00	1.13	1.29	1.50	1.80
0.4	0.53	0.57	0.62	0.67	0.73	0.80	0.89	1.00	1.14	1.33	1.60
0.6	0.47	0.50	0.54	0.58	0.64	0.70	0.78	0.88	1.00	1.17	1.40
0.8	0.40	0.43	0.46	0.50	0.55	0.60	0.67	0.75	0.86	1.00	1.20
1	0.33	0.36	0.38	0.42	0.45	0.50	0.56	0.63	0.71	0.83	1.00

7. Optimal Taxation with Many Consumers

Suppose that we do not have a single consumer or that all consumers do not have the same preferences. In choosing the optimal tax structure we must make tradeoffs between the gains to different consumers. To do this requires some explicit weighting of the utility of the different individuals. This weighting is done through the use of a social welfare function, $W(U_1, U_2, \ldots, U_n)$ where the arguments m the social welfare function are the utility levels of the individuals in society. We assume that $\partial W/\partial U_i > 0$ and $\partial^2 W/\partial U_i^2 < 0$ for all individuals *i*. Note that use of the welfare function requires we have cardinal utility so we can compare across individuals. The problem now becomes

$$\underset{\tau_1,\tau_2,\ldots,\tau_n}{Maximize}\ W(V_1, V_2, \ldots, V_n)$$
$$s.t.\ \sum_{k=1}^{K} \tau_k \left(\sum_{i=1}^{n} x_k^i(\underline{q}) \right) \geq R \qquad (37)$$

where x_k^i is the demand for x_k by individual *i.* Letting λ be the LaGrange multiplier, W_i, equal $\partial W/\partial U_i$, the marginal social welfare of a gain in utility to individual i. Then using Roy's identity to substitute for $\partial V_i/\partial q_k = -\alpha^i x_k^i$ where α^i is the marginal utility of income, the first order condition for (37) is

$$\sum_{i=1}^{n} W_i \alpha^i x_j^i = \lambda \left[X_j + \sum_{k=1}^{K} \tau_k \left(\sum_{i=1}^{n} \frac{\partial x_k^i}{\partial q_j} \right) \right] \ for\ all\ j. \qquad (38)$$

The notation X_k in (38) is the total demand for good k. Let β^i equal $W_i \alpha^i$, the marginal social value of income to individual *i.* Then after apply the Slutsky decomposition and the symmetry of the Slutsky matrix and then rearranging (38) we obtain:

$$\sum_{k=1}^{K}\frac{\tau_K}{q_k}\left[\sum_{i=1}^{n}\eta_{jk}^{i}\left(\frac{x_j^i}{X_j}\right)\right] = -\underbrace{\left[1-\sum_{i=1}^{n}\left(\frac{\beta^i}{\lambda}+\sum_{k=1}^{K}\tau_k\frac{\partial x_k^i}{\partial y^i}\right)\left(\frac{x_j^i}{X_j}\right)\right]}_{\theta_j} \tag{39}$$

The left side is similar to the traditional elasticity rule for a single consumer and can be interpreted as the compensated reduction in demand for good *j.* Except in this case the sum of each individual's utility is weighted by their consumption of x_j relative to total consumption of x_j, X_j. Note that unlike the single consumer case we do not have the sum of the elasticities equal to a constant as the right side (θ_j) depends on the consumption patterns of individuals. The magnitude of θ_j, the reduction in compensated demand, is lower:

1) the more the good is consumed by individuals with a high marginal social valuation;
2) the more the good is consumed by households with a high marginal propensity to consume taxed goods;

The reduction in compensated demand will be equal for all goods only if:

1) the net social marginal valuation of income is the same for all individuals ($\beta^i/\lambda + \sum_{k=1}^{K}\tau_k \frac{\partial x_k^i}{\partial y^i}$); or

2) $\frac{x_k^i}{X_k}$ is the same for all commodities.