ECO 751: Public Economics
Fall 2007
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Lecture 1: Normative Analysis and Welfare Economics

1. The Terminology of Welfare Economics

Efficient (Pareto Optimal, Pareto Efficient, First Best)

- Identifies a set of allocations.
- Used independently of concerns for income distribution.
- Assumes the full use of policy tools (i.e. lump sum taxation is available and full information) implying that any desired allocation can be obtained.
- This concept is used to highlight allocation problems arising from market failures (public goods, externalities) and distorting taxes.

Social Welfare Maximizing (Social Welfare Function)

- Identifies a unique efficient (first or second best) allocation (or limited number) based on social welfare function that represents distributional concerns.
- Requires cardinal utility and interpersonal utility comparisons.

The concept of a social welfare function addresses issues of equity by making the objective to obtain an allocation that addresses distributional issues. For example:

Utilitarian: $U_1 + U_2$ Government allocates goods to maximize total utility to society (Bentham, Mills)

Rawlsian: Maximize (Minimum (U_1,U_2)) Government allocates goods to maximize the well-being of the worst off individual. (John Rawls, <u>A Theory of Justice</u>).

Second Best (Second Best Efficient)

- Identifies the preferred allocation or set of allocations when only a limited number of policy tools available (no lump sum taxation or transfers) meaning only a limited number of allocations can be obtained.
- Second best allocations generally maximizes a social welfare function and is not efficient.
- Used when examining and designing taxation and transfer programs.

The differences in these concepts and the appropriate concept to use depends on:

- Concerns for Income Distribution and Equity Issues
- Policy Tools available
- Information

2. The Definition of Efficiency

We offer two alternative definitions of efficiency. One definition will prove useful in thinking about efficiency in terms of the allocations of goods between individuals (welfare economics). The other definition is useful in terms of thinking about the efficient level of production for a good in the economy (supply and demand).

Consider two alternative but equivalent definitions:

1. An allocation is said to be efficient if there is no means of reallocating goods or altering production to make some person(s) better off without making someone else worse off. (The only way someone can be made better off is by making someone else worse off. We can not find any voluntary trades to be made.)

2. An allocation is said to be efficient if it maximizes net benefit to society. (All goods are produced to quantity where Marginal Social Benefit = Marginal Social Cost).

5. *Efficiency and Welfare Economics*

Use of the supply and demand framework and net benefit analysis, enables us to understand the *efficient* level of product but it does not say anything about a number of other conditions necessary for efficiency. In particular, it does not address the issue of how to allocate goods among individuals nor how goods should be produced, that is, how resources should be allocated among firms. Use of "welfare economics" focuses on the efficiency and allocation.

An efficient allocation of goods in the economy requires that three separate efficiency conditions must be satisfied:

1. **Allocative efficiency** requires that goods in the economy that have been produced are efficiently allocated among consumers.
2. **Productive efficiency** requires that we use our resources to produce the goods that consumers desire.
3. **Technical efficiency** requires we use our resources in production appropriately.

A. Allocative Efficiency in Consumption

We wish to look at the efficiency conditions in another way -- a way that focuses on the notion of trade among consumers (or producers). We begin with an example of an **exchange** economy in which no goods are produced. Instead, the problem that we focus on is how to allocate the goods that we have produced. Again, we apply our concept of finding possible trades among the individuals to see if the commodities being produced are efficiently allocated.

B. Technical Efficiency

What determines an efficient allocation of our resources in production? What industries should be labor intensive and what should be capital intensive?

Definition: Marginal Product (MP) The marginal product of input in production, say labor, is simply defined to be the addition in output from an increase in one unit of the input. In other words, how much additional output is gained from hiring another unit of the input holding the use of other inputs constant.

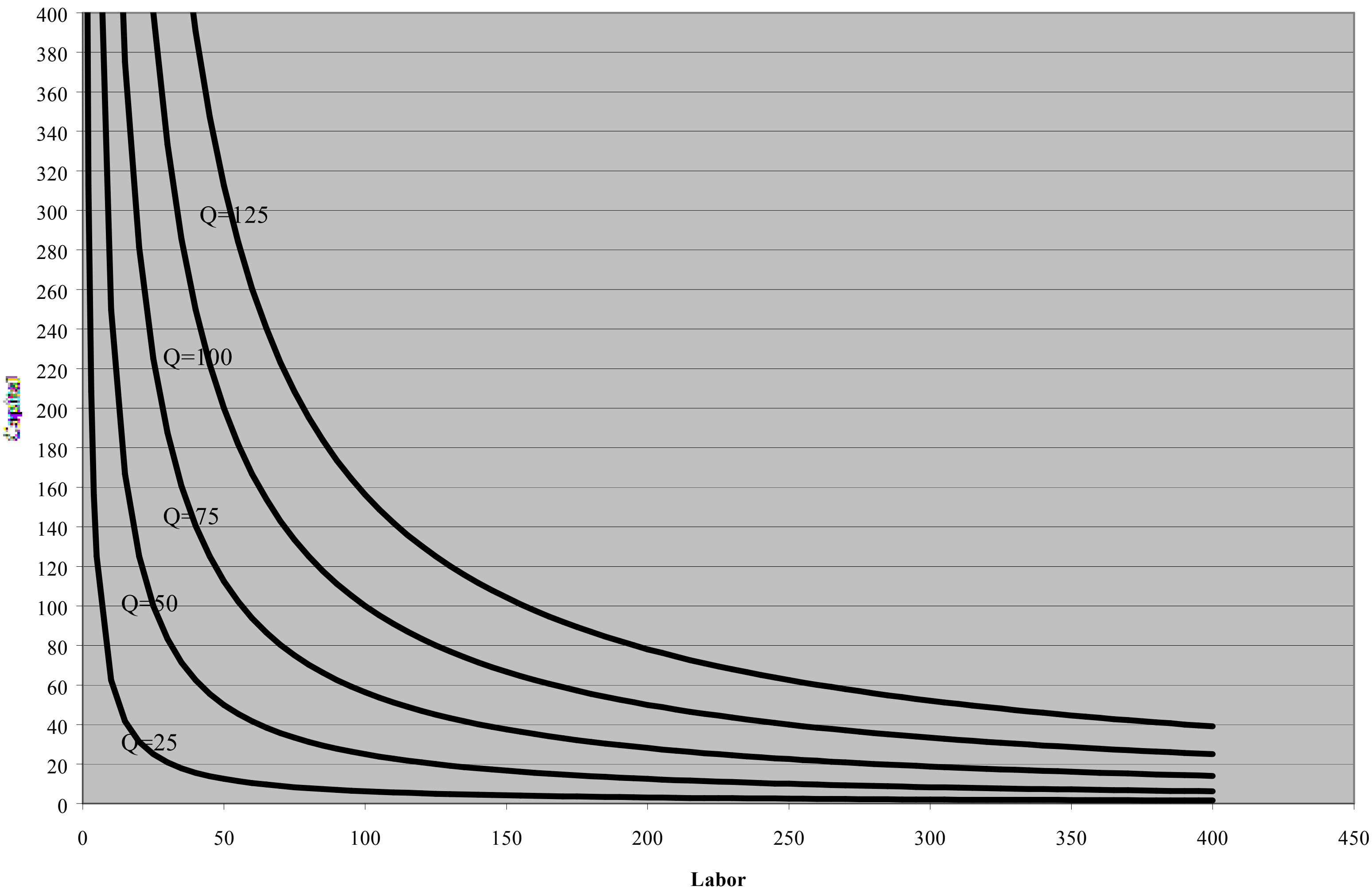
Definition: Marginal Rate of Technical Substitution ($MRTS_{LK} = MP_L/MP_K$). The marginal rate of substitution is the ratio of the productivity between two inputs, labor and capital for example.

An allocation of resources (capital and labor) will be technically efficient only when the marginal rates of technical substitution or, equivalently, the ratios of marginal products is the same in all firms that use the two inputs. It is relative marginal productivity that matters.

Graphically, we can represent the production process through use of *isoquants*. All combinations of inputs (capital and labor) that can produce the same level of output are on the same isoquant. The slope of the isoquant is the MRTS.

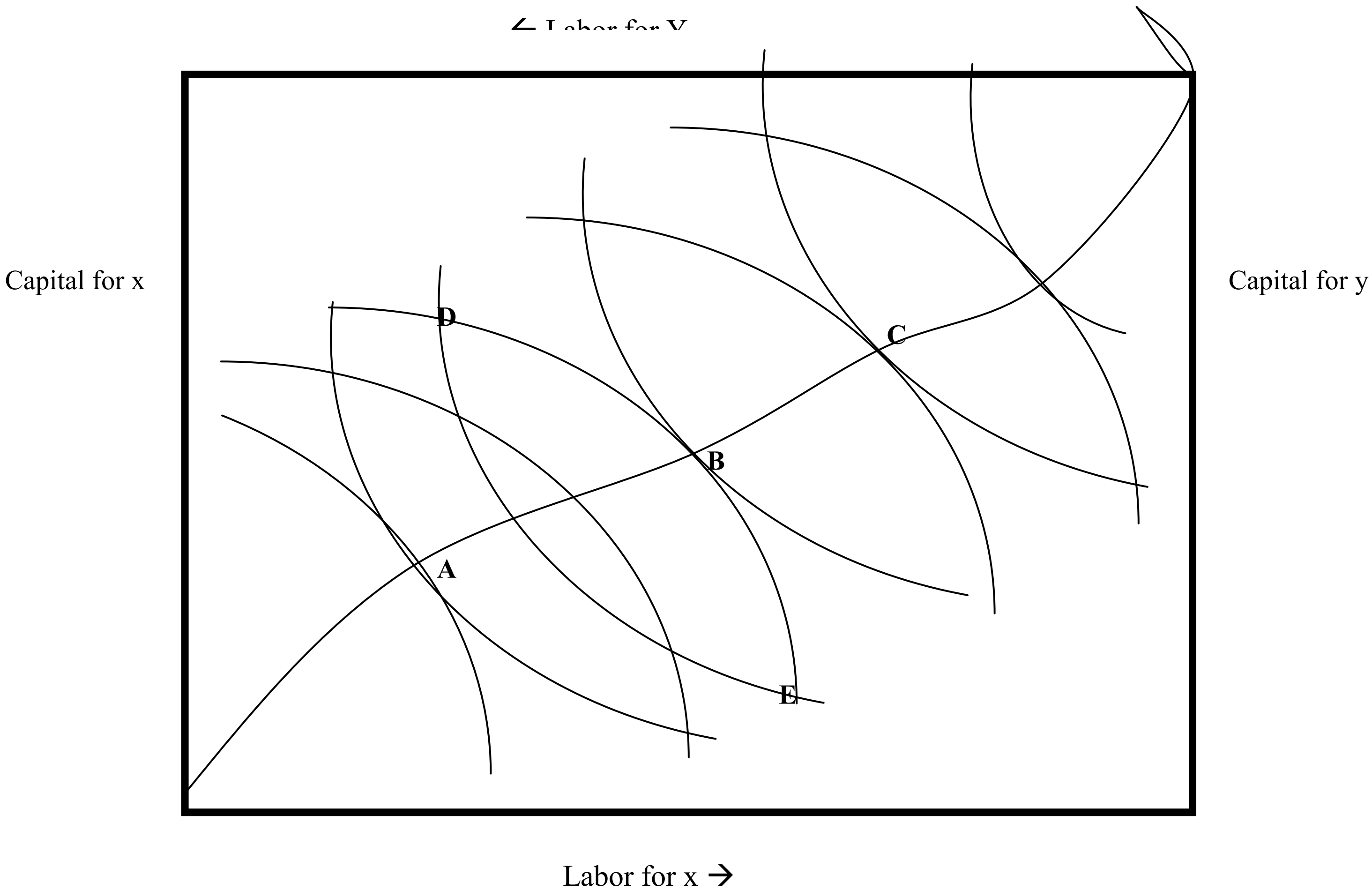
In *Figure 4* we have the isoquants for Q = 25, Q = 50, Q = 100, and Q = 200 for a firm. From *Figure 4* we can see that with 50 units of labor we need 100 units of capital to produce Q = 75.

Figure 4: An Isoquant Representation of Production



Then we can illustrate the efficiency allocation of resource (capital and labor) in production between two goods (x and y) in an Edgeworth-Boley box as well. In *Figure 5* allocations A, B, and C where the isoquants are tanquent are efficient as the slopes of the isoquants are the marginal rates of technical substituion, (MRTS) then at these points $MRTS^x = MRTS^y$. At D and E the MRTS are not equal and these are inefficient allocations.

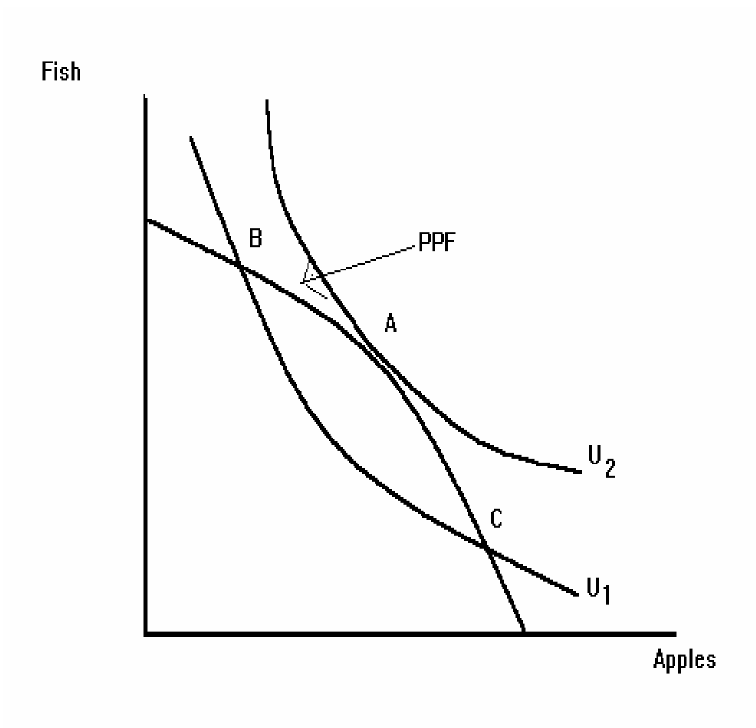
Figure 5: The Efficient Allocation of Resources in Production



C. Efficient Levels of Production

What determines the efficient level of production of two goods x and y? Consider a simple example of a single consumer/producer (Robinson Crusoe) who both fishes and picks apples and consumes all that he harvests.

Figure 6: Efficient Production



Definition: The marginal rate of transformation (MRT) is the slope of the production possibility curve and gives the amount of good y that must be sacrificed for a unit of x ($\Delta Q_y/\Delta Q_x$) or the opportunity cost of good x.

In Figure 6, the production possibility curve for apples and fish for Robinson is represented. Also represented are his indifference curves representing his tastes for apples and fish. Clearly, Robinson should be allocating his time to obtain allocation A. Note that allocation A, the indifference curve and the production possibility curve are tangent meaning their slopes are equal.

For an allocation of production between goods x and y to be efficient it must be the case that the marginal rate of transformation must equal the marginal rate of substitution for all individuals who consume the two goods, MRT = MRS.

4. A Formal Derivation of the Efficiency Conditions

A. Efficiency in a Trade (Endowment) Economy

We now wish to formally derive the efficiency conditions using the technique of constrained optimization.

Use of this approach gives us some intuition regarding the efficient allocation of resources since a critical element in determining an efficient allocation is an understanding that resources are limited and desires are not. We begin with considering only the allocation of goods consumed by individuals and not allocating resources used in production. This is referred to as an "endowment" economy.

Two Good Example

Let there be two individuals (A and B) and two goods (x and y). We seek to determine an efficient allocation of goods or more precisely find the conditions that describe an efficient allocation. How could we characterize this problem?

Recall that:

- An efficient allocation requires no one person can be made better off without making anyone else worse off;
- If every good is valued by at least one person, in an efficient allocation every good will be fully consumed but no more can be consumed than exists.

We can think of the problem as choosing the allocations of goods (in this case x_A, y_A, x_B, and y_B) making someone as well off as possible while guaranteeing some level of utility for everyone else – thereby not making them worse off. In addition, we must do this given our constrained resources. Then we can think about the problem with two individuals and two goods as

$$\underset{x_A, y_A, x_B, y_B}{Maximize}\; U^A(x_A, y_A) \qquad (1)$$

s.t.

$$U^B(x_B, y_B) \geq \overline{U}^B \qquad (2a)$$

$$x_A + x_B \leq \overline{x} \qquad (2b)$$

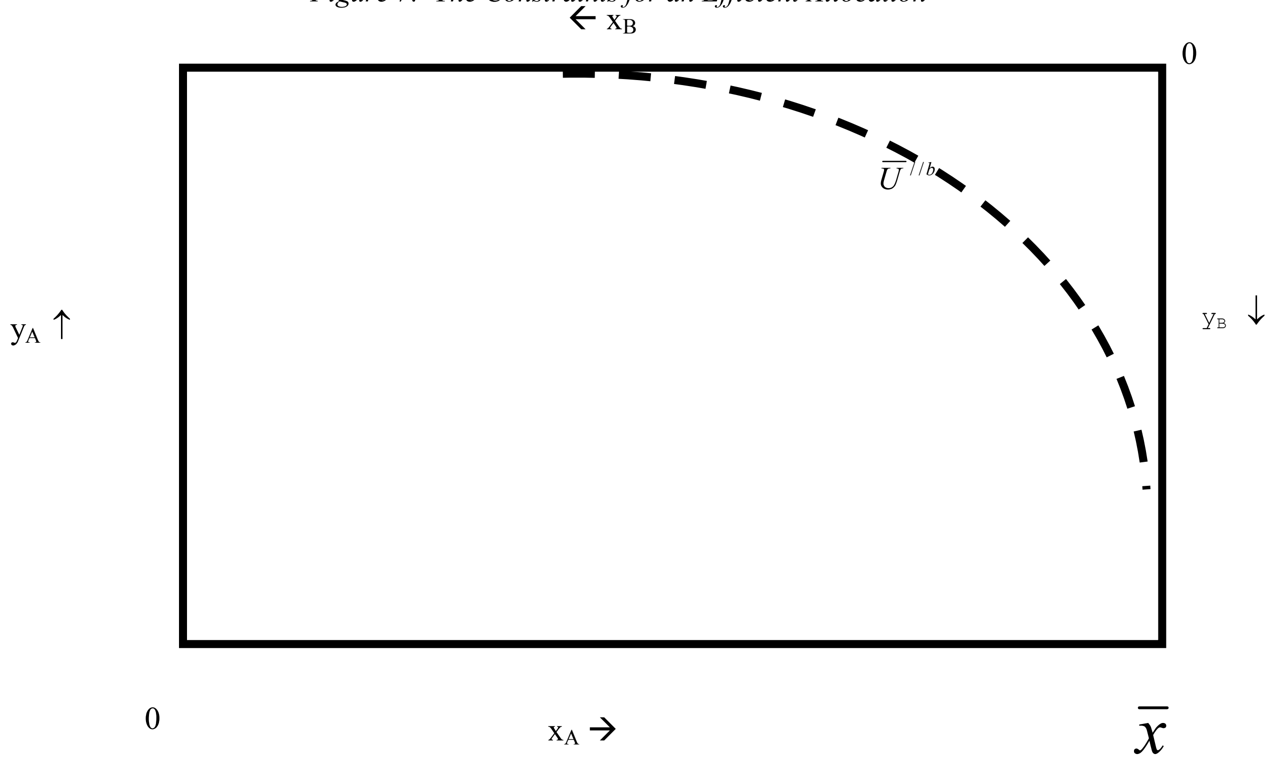
$$y_A + y_B \leq \overline{y} \qquad (2c)$$

$$x_A \geq 0, y_A \geq 0, x_B \geq 0, y_B \geq 0 \qquad (2d)$$

We seek to maximize A's utility subject to the requirement that B receives utility at least equal to $\overline{U}^B$ and that we do not consume more than our endowments of the two goods, $\overline{x}$ *and* $\overline{y}$. Before we formally solve this problem note that the constraints (2) can be depicted graphically using the Edgeworth-Boley box *(Figure 7)*. The endowment constraints (2b) and (2c) give the size of the box and (2a) gives another constraint. Condition (2d) is a technical condition that simply states that negative quantities of a good can not be consumed (no negative quadrants in *Figure 7)*.

If we assume that A's utility is always increasing in at least one of the two goods then we know that the efficient allocation must fall on the line given by the indifference curve for B that gives her $\overline{U}^B$ and within the box.

Figure 7: The Constraints for an Efficient Allocation



We now consider the more formal characterization of the problem. The constraints imposed on the problem (2) were inequality constraints. However, if we assume that at least one of the two individuals always values one of the two goods and that tastes are insatiable—utility is always increasing in at least one of the two goods. Then we have the following Lagrangean function

$$L(x_A, y_A, x_B, y_B, \lambda, \mu_x, \mu_y) = U^A(x_A, y_A) + \lambda\left[\overline{U}^B - U^B(x_B, y_B)\right] \\ + \mu_x\left[\bar{x} - \left(x_A + x_B\right)\right] + \mu_y\left[\bar{y} - \left(y_A + y_B\right)\right] \tag{3}$$

We seek to find the values for $x_A, y_A, x_B, y_B, \lambda, \mu_x, \mu_y$ that maximize this function. This, like any other function would be where the first derivatives all equal zero that is where $\partial L/\partial x_A = \partial L/\partial x_B = \partial L/\partial y_A = \partial L/\partial y_B = \partial L/\partial \lambda = \partial L/\partial \mu_x = \partial L/\partial \mu_y = 0$. However because we cannot have negative consumption of any good we have slightly different first order conditions. These conditions are referred to as the "Kuhn-Tucker" conditions and for this problem they are

$$\frac{\partial L}{\partial x_A} = \frac{\partial U^A}{\partial x_A} - \mu_x \leq 0 \; (= 0 \; if \;\; x_A > 0) \tag{4a}$$

$$\frac{\partial L}{\partial y_A} = \frac{\partial U^A}{\partial y_A} - \mu_y \leq 0 \; (= 0 \; if \;\; y_A > 0) \tag{4b}$$

$$\frac{\partial L}{\partial x_B} = \lambda \frac{\partial U^B}{\partial x_B} - \mu_x \leq 0 (= 0 \, if \; x_B > 0) \tag{4c}$$

$$\frac{\partial L}{\partial y_B} = \lambda \frac{\partial U^B}{\partial y_B} - \mu_y \le 0 (= 0 \, if \; y_B > 0) \tag{4d}$$

Alternatively these conditions are frequently expressed as

$$x_A \frac{\partial L}{\partial x_A} = x_A \left(\frac{\partial U^A}{\partial x_A} - \mu_x \right) = 0 \tag{4a'}$$

$$y_A \frac{\partial L}{\partial y_A} = y_A \left(\frac{\partial U^A}{\partial y_A} - \mu_y \right) = 0 \tag{4b'}$$

$$x_B \frac{\partial L}{\partial x_B} = x_B \left(\lambda \frac{\partial U^B}{\partial x_B} - \mu_x \right) = 0 \tag{4c'}$$

$$y_B \frac{\partial L}{\partial y_B} = y_B \left(\lambda \frac{\partial U^B}{\partial y_B} - \mu_y \right) = 0 \tag{4d'}$$

This form of expressing the conditions makes it clear that either the allocation of the good x_i, i = A, B; y_i, i = A, B is equal to zero or the bracketed term must equal zero.

Our interest will generally not be in the case of when individuals do not consume any of a good so let us consider the case when we have an "interior" solution with $x_A > 0$, $x_B > 0$, $y_A > 0$, and $y_B > 0$. (Conditions that guarantee this are that $\left.\frac{\partial U^i}{\partial x_i}\right|_{x_i \to 0} \to \infty,\ i = A, B \ and\ \left.\frac{\partial U^i}{\partial y_i}\right|_{y_i \to 0} \to \infty,\ i = A, B$.) In this case the first order conditions become

$$\frac{\partial L}{\partial x_A} = \frac{\partial U^A}{\partial x_A} - \mu_x = 0 \tag{4a''}$$

$$\frac{\partial L}{\partial y_A} = \frac{\partial U^A}{\partial y_A} - \mu_y = 0 \tag{4b''}$$

$$\frac{\partial L}{\partial x_B} = \lambda \frac{\partial U^B}{\partial x_B} - \mu_x = 0 \tag{4c''}$$

$$\frac{\partial L}{\partial y_B} = \lambda \frac{\partial U^B}{\partial y_B} - \mu_y = 0 \tag{4d''}$$

What do these conditions tell us? If we divide (4a) by (4b) and (4c) by (4d) we obtain the following

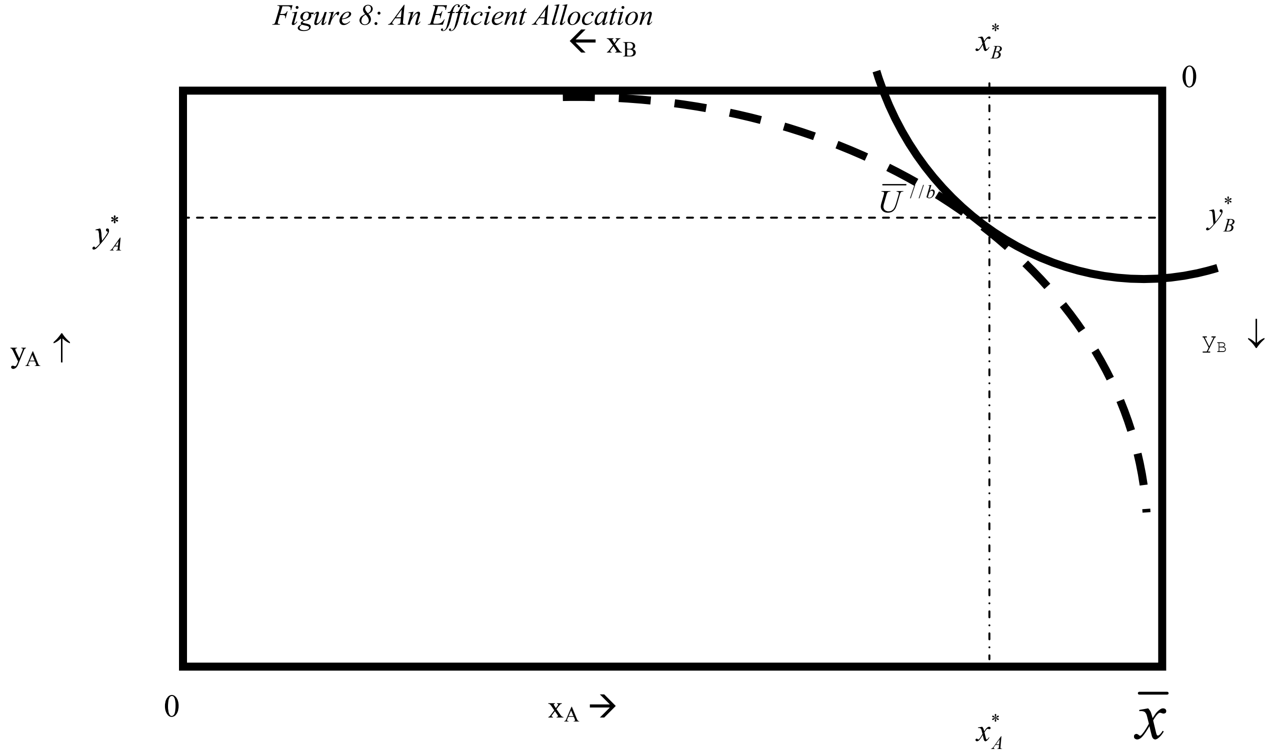
$$\frac{\frac{\partial U^A}{\partial x_A}}{\frac{\partial U^A}{\partial y_A}} = \frac{\mu_x}{\mu_y} = \frac{\frac{\partial U^B}{\partial x_B}}{\frac{\partial U^B}{\partial y_B}} \tag{5}$$

or, equivalently,

$$MRS_{xy}^{A} = \frac{\mu_x}{\mu_y} = MRS_{xy}^{B} \ , \qquad (5')$$

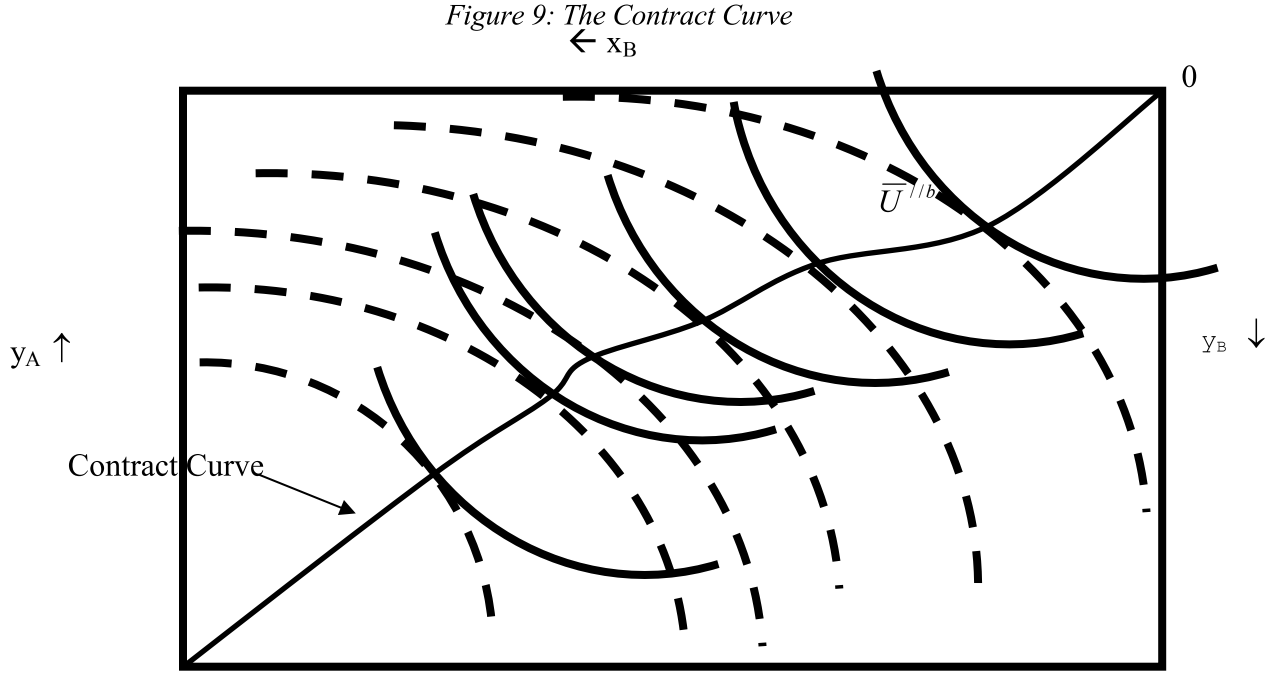
the marginal rates of substitution must be equal in an efficient allocation (when both individuals consume both goods). Graphically, then, the efficient allocation is given in *Figure 8*.

Figure 8: An Efficient Allocation



Note: We can generate the set of efficient allocations (the contract curve) by altering the minimum utility required by individual B, $\overline{U}^{B}$.

Figure 9: The Contract Curve



0 $x_A \rightarrow$ $\overline{X}$

LaGrange Multipliers

We optimize the LaGrangian function, L(.), not only with respect to the allocations of goods but also with respect to the *LaGrange multipliers*. What economic content do these multipliers have? To see this note that

$$\frac{\partial L}{\partial \overline{U}^B} = \lambda; \; \frac{\partial L}{\partial \overline{X}} = \mu_x; \; \frac{\partial L}{\partial \overline{Y}} = \mu_y.$$

The multipliers give the change in the objective function (utility of individual A) with respect to a change in the value of the constraints. Thus λ is the change in U^A from a change in U^B, $\lambda = dU^A/dU^B < 0$. A change in the endowment of x $\left(d\overline{X}\right)$ changes utility by $\mu_x d\overline{X}$ *or* $\mu_x = \dfrac{dU^A}{d\overline{X}} > 0$ with an analogous interpretation for μ_y. We refer to these multipliers as the "shadow" prices of the constraint.

B. An Example of an Endowment Economy

Let there be two goods, x and y, and two individuals, A and B. Both have Cobb-Douglas utility functions:

$$U^A = x_A^{\alpha} y_A^{1-\alpha} \text{ and } U^B = x_B^{\gamma} y_B^{1-\gamma} \tag{6}$$

There is an endowment of x, $\overline{X}$, and an endowment of y, $\overline{Y}$. Then the efficient allocation (x_A, y_A, x_B, and y_B) solves the problem

$$\begin{aligned} &\underset{x_A, y_a, x_B, y_B}{Maximize}\; U^A = x_A^{\alpha} y_A^{1-\alpha} \\ &s.t.\; U^B = x_B^{\gamma} y_B^{1-\gamma} \geq \overline{U}^B \\ &\quad x_A + x_B \leq \overline{X} \\ &\quad y_A + y_B \leq \overline{Y} \end{aligned} \tag{7}$$

The LaGrangian function L(x_A, y_A, x_B, y_B, λ, μ_x, μ_y) is

$$\begin{aligned} L(x_A, y_A, x_B, y_B, \lambda, \mu_x, \mu_y) = x_A^{\alpha} y_A^{1-\alpha} + \lambda\left(\overline{U}^B - x_B^{\gamma} y_B^{1-\gamma}\right) + \mu_x\left(\overline{X} - \left(x_A + x_B\right)\right) \\ + \mu_y\left(\overline{Y} - \left(y_A + y_B\right)\right) \end{aligned} \tag{8}$$

Then the first order conditions (assuming an interior solution – which will be the case here) are:

$$\frac{\partial L}{\partial x_A} = x_A^{\alpha-1} y_A^{1-\alpha} - \mu_x = 0, \tag{9a}$$

$$\frac{\partial L}{\partial y_A} = x_A^{\alpha} y_A^{-\alpha} - \mu_y = 0, \tag{9b}$$

$$\frac{\partial L}{\partial x_B} = \lambda x_B^{\gamma-1} y_B^{1-\gamma} - \mu_x = 0, \tag{9c}$$

$$\frac{\partial L}{\partial y_B} = \lambda x_B^{\gamma} y_B^{-\gamma} - \mu_y = 0, \tag{9d}$$

$$\frac{\partial L}{\partial \lambda} = \overline{U}^{B} - x_B^{\gamma} y_B^{1-\gamma} = 0 \tag{9e}$$

$$\frac{\partial L}{\partial \mu_x} = \overline{X} - \left(x_A + x_B\right) = 0 \tag{9f}$$

$$\frac{\partial L}{\partial \mu_y} = \overline{Y} - \left(y_A + y_B\right) = 0 \tag{9g}$$

Using (9) we can begin to solve for the efficient allocation. Dividing (9a)/(9b) and (9c)/(9d) and simplifying gives

$$\frac{\alpha}{(1-\alpha)} \frac{y_A}{x_A} = \frac{\mu_x}{\mu_y} = \frac{\gamma}{(1-\gamma)} \frac{y_B}{x_B} \tag{10}$$

where these terms are simply the marginal rates of substitution for A and B and must be equal in an efficient allocation. (10) gives the contract curve. Then we can use (9f) and (9g), endowment constraints, to substitute for x_B and y_B and obtain:

$$\frac{\alpha}{(1-\alpha)} \frac{y_A}{x_A} = \frac{\mu_x}{\mu_y} = \frac{\gamma}{(1-\gamma)} \frac{\left(\overline{Y} - y_A\right)}{\left(\overline{X} - x_A\right)} \tag{11}$$

Solving (11) for x_A gives

$$x_A = \frac{\overline{X}}{\left[\frac{\gamma}{(1-\gamma)} \frac{(1-\alpha)}{\alpha} \left(\frac{\overline{Y}}{y_A} - 1\right) + 1\right]} \tag{12}$$

(12) gives the contract curve in terms of x_A and y_A only. Given x_A and y_A from (12), x_B and y_B are determined by (9f) and (9g). To find the specific allocation that solves (7) we use the minimum utility constraint for B (9e) and substitute for x_B and y_B in it using (9f) and (9g). Then performing these substitutions gives

$$(\overline{X} - x_A)^{\gamma} (\overline{Y} - y_A)^{1-\gamma} = \overline{U}^{B} \tag{13}$$

Equations (12) and (13) give two equations and two unknowns (x_A and y_A) that can be solved to determine the efficient allocation (These are highly non-linear so they can not be solved directly).

A Numerical Example

Let $\alpha = .25$ and $\gamma = .5$ and $\overline{X} = 1$ *and* $\overline{Y} = 1$. Then (12) implies that

$$x_A = \frac{1}{\left[3\left(\frac{1}{y_A} - 1\right) + 1\right]} \tag{14}$$

Then using (14) if $y_A = .5$ we have $x_A = .25$ and if $y_A = .75$ we have $x_A = .5$ as examples.

5. *An Economy with Production*

How do you add production? Why do you add it? As we discussed efficiency not only requires an efficient *allocation* of the products that have been produced but also an efficient *production* levels of the products and *technical* efficiency in production. To understand and derive the conditions for these two aspects of efficiency we need to introduce production into the economy.

Let the model have two goods, x and y; two individuals, A and B; and two factors using in production, K and L (capital and labor). Let the production functions for the two goods be given by

$$X = F_x(k_x, l_x) \tag{15a}$$

and

$$Y = F_y(k_y, l_y) \tag{15b}$$

where k_j and l_j, are the amounts of capital and labor used in producing good j, j=x,y. We assume that production is increasing in both capital and labor, $\partial F_j/\partial k_j > 0$ and $\partial F_j/\partial l_j > 0$, j = x, y but at a decreasing rate, $\partial^2 F_j/\partial k_j^2 < 0$ and $\partial^2 F_j/\partial l_j^2 < 0$, j = x,y. We refer to $\partial F_j/\partial k_j$ as the *marginal product of capital* (MP_k^j) and $\partial F_j/\partial l_j$ as the *marginal product of labor* (MP_l^j). The assumptions that $\partial^2 F_j/\partial k_j^2 < 0$ and $\partial^2 F_j/\partial l_j^2 < 0$, j = x,y are simply the assumption that there exists *diminishing marginal products.*

With production there is no fixed endowment of the consumption goods, x and y; instead we choose how much of these to produce based on allocating a fixed endowment of capital and labor, $\overline{K}$ *and* $\overline{L}$. Then the problem now becomes:

$$\underset{x_A, y_A, x_B, y_B, l_x, k_x, l_y, k_y}{Maximize} \quad U^A(x_A, y_A) \tag{16}$$

s.t.

$$U^B(x_B, y_B) \geq \overline{U}^B \tag{17a}$$

$$F_x(l_x, k_x) \geq x_A + x_B \tag{17b}$$

$$F_y(l_y, k_y) \geq y_A + y_B \tag{17c}$$

$$\overline{L} \geq l_x + l_y \tag{17d}$$

$$\overline{K} \geq k_x + k_y \tag{17e}$$

Then given (16) and (17) we have the LaGrangian function

$$\begin{aligned} L(x_A, y_A, x_B, y_B, \lambda, \mu_x, \mu_y, \pi_l, \pi_k) = U^A(x_A, y_A) + \lambda\left(\overline{U}^B - U^B\right) + \\ \mu_x\left(F_x(l_x, k_x) - (x_A + x_B)\right) + \mu_y\left(F_y(l_y, k_y) - (y_A + y_B)\right) + \\ \pi_l(\overline{L} - (l_x + l_y)) + \pi_k\left(\overline{K} - (k_x + k_y)\right) \end{aligned} \tag{18}$$

Then the first order conditions (for an interior solution) are:

$$\frac{\partial L}{\partial x_A} = \frac{\partial U^A}{\partial x_A} - \mu_x = 0, \tag{19a}$$

$$\frac{\partial L}{\partial y_A} = \frac{\partial U^A}{\partial y_A} - \mu_y = 0, \tag{19b}$$

$$\frac{\partial L}{\partial x_B} = \lambda \frac{\partial U^B}{\partial x_B} - \mu_x = 0, \tag{19c}$$

$$\frac{\partial L}{\partial y_B} = \lambda \frac{\partial U^B}{\partial y_B} - \mu_y = 0, \tag{19d}$$

$$\frac{\partial L}{\partial l_x} = \mu_x \frac{\partial F_x}{\partial l_x} - \pi_l = 0, \tag{19e}$$

$$\frac{\partial L}{\partial k_x} = \mu_x \frac{\partial F_x}{\partial k_x} - \pi_k = 0, \tag{19f}$$

$$\frac{\partial L}{\partial l_y} = \mu_y \frac{\partial F_y}{\partial l_y} - \pi_l = 0, \tag{19g}$$

$$\frac{\partial L}{\partial k_y} = \mu_y \frac{\partial F_y}{\partial k_y} - \pi_k = 0, \tag{19h}$$

$$\frac{\partial L}{\partial \lambda} = \overline{U}^B - U^B(x_B, y_B) = 0, \tag{19i}$$

$$\frac{\partial L}{\partial \mu_x} = F_x(l_x, k_x) - (x_A + x_B) = 0, \tag{19j}$$

$$\frac{\partial L}{\partial \mu_y} = F_y(l_y, k_y) - (y_A + y_B) = 0, \tag{19k}$$

$$\frac{\partial L}{\partial \pi_l} = \overline{L} - (l_x + l_y) = 0, \tag{19l}$$

$$\frac{\partial L}{\partial \pi_k} = \overline{K} - (k_x + k_y) = 0, \tag{19m}$$

Allocative Efficiency

Equations (19a) - (19d) are identical to the conditions we found in an endowment economy and simply give the conditions necessary for an efficient allocation. Using (19a) - (19d) we obtain

$$\frac{\frac{\partial U^A}{\partial x_A}}{\frac{\partial U^A}{\partial y_A}} = \frac{\mu_x}{\mu_y} = \frac{\frac{\partial U^B}{\partial x_B}}{\frac{\partial U^B}{\partial y_B}} \tag{20}$$

or, equivalently,

$$MRS_{xy}^A = \frac{\mu_x}{\mu_y} = MRS_{xy}^B \ , \tag{20'}$$

Equations (19i) - (19m) merely insure that the constraints are not violated. B receives utility equal to his minimum requirement (and no more) (19i); consumption of the goods x and y equals the production (19j-19k) and the demand for inputs k and l equals the endowment (19l-19m).

Technical Efficiency

Then we can use (19e) - (19h) to investigate the allocation of inputs into production. Dividing (19e) by (19f) and (19g) by (19h) gives

$$\frac{\frac{\partial F_x}{\partial l_x}}{\frac{\partial F_x}{\partial k_x}} = \frac{\pi_l}{\pi_k} = \frac{\frac{\partial F_y}{\partial l_y}}{\frac{\partial F_y}{\partial k_y}} \tag{21}$$

This is the condition for *technical efficiency*. We refer to $\frac{\frac{\partial F_x}{\partial l_x}}{\frac{\partial F_x}{\partial k_x}}$ *and* $\frac{\frac{\partial F_y}{\partial l_y}}{\frac{\partial F_y}{\partial k_y}}$ as the marginal rates of technical substitution so this condition is simply

$$MRTS_{lk}^x = MRTS_{lk}^y . \tag{21'}$$

Productive Efficiency

This condition is more difficulty and less directly derived but can be obtained from the first order conditions (19). We begin by noting that changes in production of x and y (dx and dy) depend on the production functions for x and y, ($F_x(l_x,k_x)$ and $F_y(l_y,k_y)$), and the changes in capital and labor used in producing the two goods, (dk_x, dl_x, dk_y, *and* dl_y). Then differentiating the production functions with respect to capital and labor gives

$$dx = \frac{\partial F_x}{\partial l_x} dl_x + \frac{\partial F_x}{\partial k_x} dk_x \tag{22a}$$

$$dy = \frac{\partial F_y}{\partial l_y} dl_y + \frac{\partial F_y}{\partial k_y} dk_y \tag{22b}$$

Note that given the fixed endowments of capital and labor it is the case that dl_y = -dl_x and dk_y = -dk_y. Then (22b) becomes

$$dy = -\frac{\partial F_y}{\partial l_y} dl_x + -\frac{\partial F_y}{\partial k_y} dk_x \qquad \text{(22b')}$$

Using (19e) - (19h) we have

$$\frac{\partial F_x}{\partial l_x} = \frac{\pi_l}{\mu_x}; \frac{\partial F_x}{\partial k_x} = \frac{\pi_k}{\mu_x}; \frac{\partial F_y}{\partial l_y} = \frac{\pi_l}{\mu_y}; \frac{\partial F_y}{\partial k_y} = \frac{\pi_k}{\mu_y}. \qquad \text{(23)}$$

Then using (23) to substitute into (22a) gives

$$dx = \frac{\pi_l}{\mu_x} dl_x + \frac{\pi_k}{\mu_x} dk_x = \frac{1}{\mu_x}\left(\pi_l dl_x + \pi_k dk_x\right) \qquad \text{(24a)}$$

and using (23) to substitute into (22b') gives

$$dy = -\frac{\pi_l}{\mu_y} dl_x + -\frac{\pi_k}{\mu_y} dk_x = -\frac{1}{\mu_y}\left(\pi_l dl_x + \pi_k dk_x\right) \qquad \text{(24b)}$$

then (24b)/(24a) gives

$$-\frac{dy}{dx} = \frac{\mu_x}{\mu_y} \qquad \text{(25)}$$

which is referred to as the marginal rate of transformation (negative sign makes this a positive number. Then from (20) we have

$$\frac{\dfrac{\partial U^A}{\partial x_A}}{\dfrac{\partial U^A}{\partial y_A}} = \frac{\dfrac{\partial U^B}{\partial x_B}}{\dfrac{\partial U^B}{\partial y_B}} = \frac{\mu_x}{\mu_y} = -\frac{dy}{dx} \qquad \text{(26)}$$

or

$$MRS_{xy}^A = MRS_{xy}^B = MRT_{xy} \qquad \text{(26')}$$