

The Dual Approach and its Uses in Public Economics

1. *Introduction*

The traditional approach to considering the problem faced by a consumer is to maximize the consumer's utility by choosing his consumption subject to given prices and income. However, in public economics we are not interested in what the consumer does given prices but on how changes in prices and income, through changes in public policies, will affect consumer well-being. This means that we are interested in the consumer's well-being (utility) as a function of prices and income given the consumer chooses his consumption bundle to maximize utility. This is known as the *indirect utility function*.

In addition, we are interested in assessing the impact of alternative policies on consumer well-being. To do this, we wish to see the income equivalent of any change in policy, that is, how much income could be taken away or how much would need to be given to keep the consumer unaffected by the policy. We discuss these concepts, compensating and equivalent variation, later. The formulation of these measures, however, requires an alternative interpretation of the consumer's problem, minimizing costs subject to a minimum utility constraint and prices. This is the "*dual*" approach and yields the expenditure function and compensated demand curves.

2. *The Consumer's Direct Problem*

We generally depict the consumer as solving the problem:

$$\underset{x,y}{\text{Maximize}} U(x, y) \tag{1}$$

$$s. t. p_x x + p_y y = I \tag{2}$$

Then the solution gives the demand for x and y as a function of prices, p_x and p_y , and income, I.

$$x = x(p_x, p_y, I) \tag{3a}$$

and

$$y = y(p_y, p_x, I). \tag{3b}$$

Then substituting for x and y in the direct utility function allows us to express utility as a function of prices and income:

$$U(x(p_x, p_y, I), y(p_y, p_x, I)) = V(p_x, p_y, I) \tag{4}$$

where $V(p_x, p_y, I)$ is the indirect utility function with

$$\frac{\partial V}{\partial P_x} \leq 0, \frac{\partial V}{\partial P_y} \leq 0, \frac{\partial V}{\partial I} \geq 0. \tag{5}$$

The indirect utility function gives the consumer's utility as a function of prices and income. It assumes the consumer maximizes utility subject to his budget. If the government can affect the consumer's budget

(through taxes and transfers) but not his choice given the budget, then the indirect utility function should be used to analyze the effects of the government programs.

2. The Dual

Consider instead the problem of the consumer obtaining a given utility level for the least amount of income given prices. The problem is then:

$$\underset{x,y}{\text{Minimize}} \quad p_x x + p_y y \quad (6)$$

$$\text{s.t.} \quad U(x, y) \geq \bar{U} \quad (7)$$

The solutions, x^* and y^* , will depend on p_x , p_y , and \bar{U} . Then the demands for x and y are given by

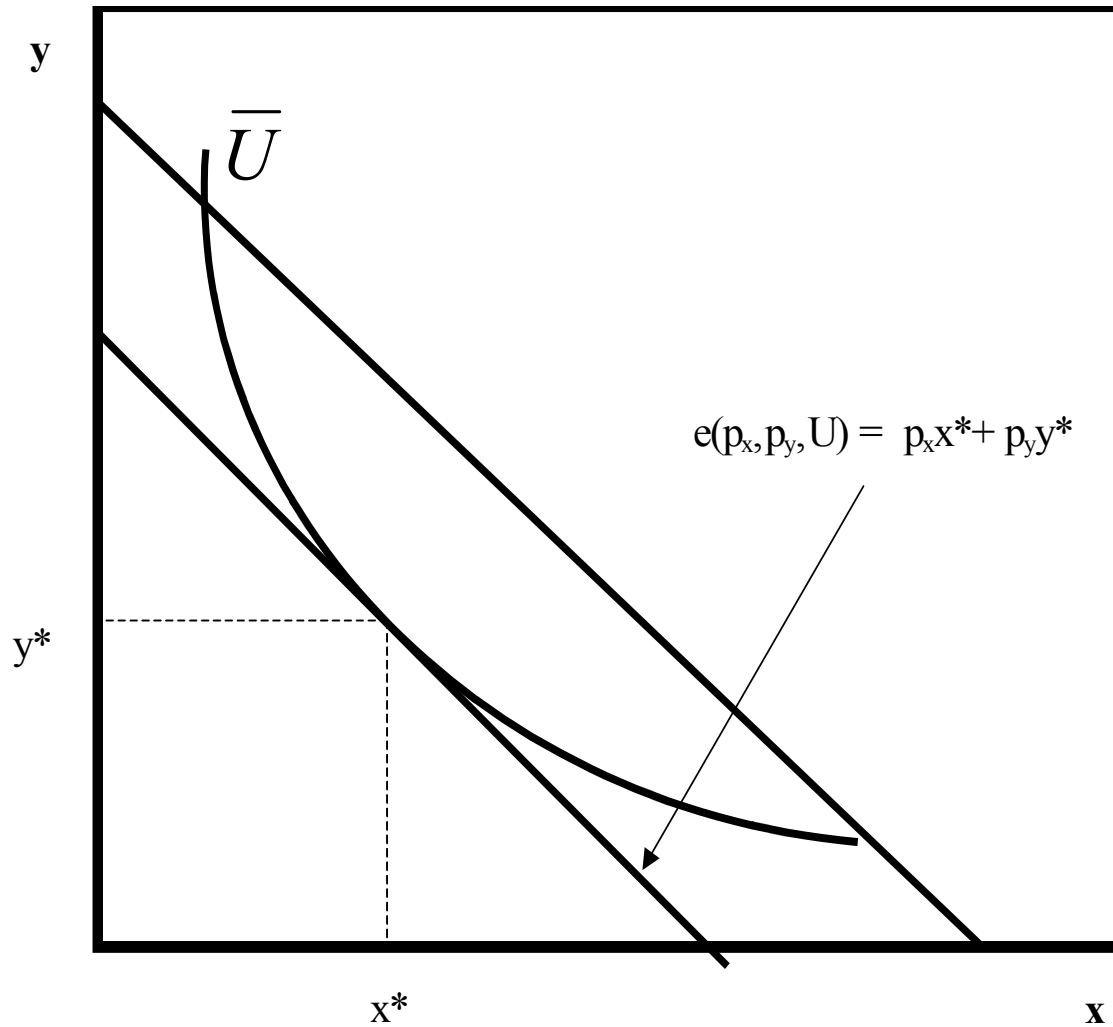
$$x = h_x(p_x, p_y, \bar{U}) \quad (8a)$$

and

$$y = h_y(p_x, p_y, \bar{U}), \quad (8b)$$

the Hicksian or compensated demand curves.

The consumer's problem is illustrated in *Figure 1*.



The expenditures needed to obtain U depend on P_x , P_y , and U with this relationship denoted by the expenditure function:

$$e(p_x, p_y, \bar{U}) = P_x h_x(P_x, P_y, \bar{U}) + P_y h_y(P_x, P_y, \bar{U}) \quad (9)$$

then $\frac{\partial e}{\partial P_x} \geq 0$, $\frac{\partial e}{\partial p_y} \geq 0$, and $\frac{\partial e}{\partial U} > 0$.

3. Some Useful Relationships

$$V(\theta p_x, \theta p_y, \theta I) = V(p_x, p_y, I) \quad \theta > 0. \quad (10)$$

The indirect utility function is homogeneous of degree one. Doubling prices and income does not change utility.

$$e(\theta p_x, \theta p_y, U) = \theta e(p_x, p_y, U), \theta > 0 \quad (11)$$

The expenditure function is homogeneous of degree one in prices. Doubling prices doubles expenditures.

$$h_x(\theta p_x, \theta p_y, U) = h_x(p_x, p_y, U) \quad (12)$$

The Hicksian demand functions are homogeneous of degree zero in prices. Doubling prices of goods will not affect compensated demand.

$$h_x(p_x, p_y, U) = x(p_x, p_y, e(p_x, p_y, U)) \quad (13)$$

The Hicksian demand at prices p_x, p_y , and utility U must equal the Marshallian demand at prices p_x, p_y , and the income needed to get utility U $e(p_x, p_y, U)$.

$$\frac{\partial e}{\partial p_x} = h_x(p_x, p_y, U) \quad (14)$$

This is known as Shephard's Lemma. The derivative of the expenditure function with respect to the price of the good is equal to the demand for that good. We can easily derive this by noting that since

$$e(p_x, p_y, U) = p_x h_x(p_x, p_y, U) + p_y h_y(p_x, p_y, U) \quad (15)$$

Then differentiating with respect to p_x gives

$$\frac{\partial e}{\partial p_x} = h_x + p_x \frac{\partial h_x}{\partial p_x} + p_y \frac{\partial h_y}{\partial p_x} \quad (16)$$

but by the envelope theorem we have $p_x \frac{\partial h_x}{\partial p_x} + p_y \frac{\partial h_y}{\partial p_x} = 0$ giving (14). In addition, we have

$$x(p_x, p_y, I) = - \frac{\frac{\partial V(p_x, p_y, I)}{\partial p_x}}{\frac{\partial V(p_x, p_y, I)}{\partial I}} \quad (17)$$

This is Roy's identity. The derivative of the indirect utility function with respect to price divided by the derivative with respect to income gives the Marshallian demand. To see this note that we can use the following relationship

$$V(p_x, p_y, e(p_x, p_y, \bar{U})) = \bar{U} \quad (18)$$

Then differentiating with respect to p_x gives

$$\frac{\partial V}{\partial p_x} + \frac{\partial V}{\partial I} \frac{\partial e}{\partial p_x} = 0. \quad (19)$$

Then using Shephard's Lemma (14) we can substitute for $h_x(p_x, p_y, U)$ for $\frac{\partial e(p_x, p_y, U)}{\partial p_x}$ and the fact that $h_x(p_x, p_y, U) = x(p_x, p_y, I)$ gives (17). Finally, we have

$$\frac{\partial h_x}{\partial p_x} = \frac{\partial x}{\partial p_x} + x \bullet \frac{\partial x}{\partial I} \quad (20a)$$

and

$$\frac{\partial h_x}{\partial p_y} = \frac{\partial x}{\partial p_y} + y \bullet \frac{\partial x}{\partial I} \quad (20b)$$

This is referred to as the Slutsky decomposition. We derive it by differentiating (13) by p_x to give

$$\frac{\partial h_x}{\partial p_x} = \frac{\partial x}{\partial p_x} + \frac{\partial x}{\partial I} \frac{\partial e}{\partial p_x} \quad (21)$$

But by Shephard's Lemma (14) we have $\frac{\partial e}{\partial p_x} = h_x(p_x, p_y, U)$ so substituting this into (21) gives (20a).

4. Some Examples

Below are specific utility functions with their associated demands, indirect utility functions, and expenditure functions.

Example 1: Cobb-Douglas Function ($U = x^\alpha y^{1-\alpha}$)

Marshallian Demand Functions:

$$x(p_x, p_y, I) = \frac{\alpha}{p_x} I \quad (22a)$$

and

$$y(p_x, p_y, I) = \frac{(1-\alpha)}{p_y} I \quad (22b)$$

Derivation of the Indirect Utility Function:

We have $U(x(p_x, p_y, I), y(p_x, p_y, I)) = V(p_x, p_y, I)$ which gives

$$V(p_x, p_y, I) = \left(\frac{\alpha}{p_x} I \right)^\alpha \left(\frac{(1-\alpha)}{p_y} I \right)^{1-\alpha} = \frac{\alpha^\alpha (1-\alpha)^{(1-\alpha)}}{p_x^\alpha p_y^{1-\alpha}} I \quad (23)$$

The Expenditure Function

Inverting $V(p_x, p_y, I)$, (23), gives $e(p_x, U)$, the expenditure function, since $V(p_x, p_y, e(p_x, p_y, U)) = U$. Then

letting $I = e(p_x, p_y, U)$ in (23) and letting $V(p_x, p_y, I) = U$ and solving for I gives

$$e(p_x, p_y, U) = \frac{p_x^\alpha p_y^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{(1-\alpha)}} U \quad (24)$$

The Hicksian Demand Functions

Differentiate the expenditure function with respect to prices to obtain Hicksian demand functions:

$$h_x(p_x, p_y, U) = \left[\frac{\alpha}{(1-\alpha)} \right]^{1-\alpha} \left(\frac{p_y}{p_x} \right)^{1-\alpha} U \quad (25a)$$

and

$$h_y(p_x, p_y, U) = \left[\frac{(1-\alpha)}{\alpha} \right]^\alpha \left(\frac{p_x}{p_y} \right)^\alpha U \quad (25b)$$

Example 2: Stone-Geary Utility Function $U = (x - \underline{x})^\alpha (y - \underline{y})^{1-\alpha}$

Marshallian Demand Equations:

$$x(p_x, p_y, I) = \frac{\alpha}{p_x} (I - p_x \underline{x} - p_y \underline{y}) + \underline{x} \quad (26a)$$

and

$$y(p_x, p_y, I) = \frac{(1-\alpha)}{p_y} (I - p_x \underline{x} - p_y \underline{y}) + \underline{y} \quad (26b)$$

Derivation of the Indirect Utility Function:

We have $U(x(p_x, p_y, I), y(p_x, p_y, I)) = V(p_x, p_y, I)$ which gives

$$V(p_x, p_y, I) = \frac{\alpha^\alpha (1-\alpha)^{(1-\alpha)}}{p_x^\alpha p_y^{1-\alpha}} (I - p_x \underline{x} - p_y \underline{y}) \quad (27)$$

The Expenditure Function

$$e(p_x, p_y, U) = \frac{p_x^\alpha p_y^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{(1-\alpha)}} U + p_x \underline{x} + p_y \underline{y} \quad (28)$$

The Hicksian Demand Functions

Differentiate the expenditure function with respect to prices to obtain Hicksian demand functions:

$$h_x(p_x, p_y, U) = \left[\frac{\alpha}{(1-\alpha)} \right]^{1-\alpha} \left(\frac{p_y}{p_x} \right)^{1-\alpha} U + \underline{x} \quad (29a)$$

and

$$h_y(p_x, p_y, U) = \left[\frac{(1-\alpha)}{\alpha} \right]^\alpha \left(\frac{p_x}{p_y} \right)^\alpha U + \underline{y} \quad (29b)$$

Example 3: The CES (Constant Elasticity of Substitution) Function $U = A[\alpha x^{-\rho} + (1-\alpha)y^{-\rho}]^{-1/\rho}$, $\rho > -1$

Marshallian Demand Equations

$$x(p_x, p_y, I) = \frac{I}{p_x \left[1 + \left(\frac{p_y}{p_x} \right)^{\frac{\rho}{1+\rho}} \left(\frac{1-\alpha}{\alpha} \right)^{\frac{1}{1+\rho}} \right]} \quad (30a)$$

and

$$y(p_x, p_y, I) = \frac{I}{p_y \left[1 + \left(\frac{p_x}{p_y} \right)^{\frac{\rho}{1+\rho}} \left(\frac{\alpha}{1-\alpha} \right)^{\frac{1}{1+\rho}} \right]} \quad (30b)$$

Indirect Utility Function

$$V(p_x, p_y, I) = A \left[\alpha p_x^{-\rho} \left[1 + \left(\frac{p_y}{p_x} \right)^{\frac{\rho}{1+\rho}} \left(\frac{1-\alpha}{\alpha} \right)^{\frac{1}{1+\rho}} \right]^{-\rho} + (1-\alpha) p_y^{-\rho} \left[1 + \left(\frac{p_x}{p_y} \right)^{\frac{\rho}{1+\rho}} \left(\frac{1-\alpha}{\alpha} \right)^{\frac{1}{1+\rho}} \right]^{-\rho} \right]^{-1/\rho} I \quad (31)$$

Expenditure Function

$$e(p_x, p_y, U) = \frac{1}{A} \left[\alpha p_x^{-\rho} \left[1 + \left(\frac{p_y}{p_x} \right)^{\frac{\rho}{1+\rho}} \left(\frac{1-\alpha}{\alpha} \right)^{\frac{1}{1+\rho}} \right]^{-\rho} + (1-\alpha) p_y^{-\rho} \left[1 + \left(\frac{p_x}{p_y} \right)^{\frac{\rho}{1+\rho}} \left(\frac{1-\alpha}{\alpha} \right)^{\frac{1}{1+\rho}} \right]^{-\rho} \right]^{1/\rho} U \quad (32)$$

$$h_x(p_x, p_y, U) = \frac{1}{A} \left[\alpha + (1-\alpha) \left(\frac{\alpha}{1-\alpha} \right)^{\frac{\rho}{\rho+1}} \left(\frac{p_y}{p_x} \right)^{\frac{\rho}{\rho+1}} \right]^{1/\rho} U \quad (33)$$